Solutions to Assignment 11

1. Write the Taylor Series with remainder term of the following functions.

(2 marks) (a) $\sin(x)/\cos(x)$ till 4 terms.

Solution: We calculate the derivatives

$$\frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)^2}$$
$$\frac{d}{dx}\frac{1}{\cos(x)^2} = \frac{2\sin(x)}{\cos(x)^3}$$
$$\frac{d}{dx}\frac{2\sin(x)}{\cos(x)^3} = \frac{2+4\sin(x)^2}{\cos(x)^4}$$

Hence the Taylor Series is

$$\frac{\sin(x)}{\cos(x)} = 0 + 1\cdot x + 0\cdot\frac{x^2}{2} + 2\cdot\frac{x^3}{3!} + gx^3$$

where g is $o(x^0)$ at 0 (in other words, g is continuous and vanishes at 0).

(2 marks) (b) $\exp(x)/(1+\exp(x))$ till 4 terms.

Solution: We calculate the derivatives

$$\frac{d}{dx}\frac{\exp(x)}{1+\exp(x)} = \frac{\exp(x)}{(1+\exp(x))^2}$$
$$\frac{d}{dx}\frac{\exp(x)}{(1+\exp(x))^2} = \frac{\exp(x)(1-\exp(x))}{(1+\exp(x))^3}$$
$$\frac{d}{dx}\frac{\exp(x)(1-\exp(x))}{(1+\exp(x))^3} = \frac{\exp(x)(1-4\exp(x)+\exp(x)^2)}{(1+\exp(x))^4}$$

Hence the Taylor Series is

$$\frac{\exp(x)}{1+\exp(x)} = \frac{1}{2} + \frac{1}{4}x + 0x^2 - \frac{1}{8}\frac{x^3}{6} + gx^3$$

where g is $o(x^0)$ at 0 (in other words, g is continuous and vanishes at 0).

(2 marks) (c) $\sin(|x|^3)$ (as many terms as possible).

Solution: We calculate the derivatives

$$\frac{d}{dx}\sin(|x|^3) = \cos(|x|^3)\cdot(3x|x|)$$
$$\frac{d^2}{dx^2}\sin(|x|^3) = -\sin(|x|^3)\cdot(3x|x|)^2 + \cos(|x|^3)\cdot(6|x|)$$

No further derivatives are possible since the last function is not differentiable. Hence the Taylor Series is

$$\sin(|x|^3) = 0 + 0x + 0\cdot\frac{x^2}{2} + gx^2$$

where g is $o(x^0)$ at 0 (in other words, g is continuous and vanishes at 0).

(2 marks) (d) Given continuously differentiable functions f and g such that $f(0) = 1$, $g(0) = 0$, $df/dx = g$ and $dg/dx = f$. Write the Taylor series of f and g and try to recognise them assuming that they are determined by the Taylor series.

Solution: We calculate

$$\frac{df}{dx}(0) = g(0) \qquad = 0$$
$$\frac{d^2f}{dx^2}(0) = f(0) \qquad = 1$$
$$\frac{d^3f}{dx^3}(0) = g(0) \qquad = 0$$

and so on. In other words,

$$\frac{d^kf}{dx^k}(0) = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Similarly,

$$\frac{d^kg}{dx^k}(0) = \begin{cases} 0 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$$

It follows that the Taylor series are

$$f(x) = 1 + \frac{x^2}{2} + \cdots + \frac{x^{2k}}{(2k)!} + o(x^{2k})$$
$$g(x) = x + \frac{x^3}{3!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+1})$$

These are the series for $\cosh(x)$ and $\sinh(x)$ respectively. Since we are *given* that f and g are determined by their Taylor series, this gives us f and g as cosh and sinh respectively.

Note that this is *not* automatic. One needs to *prove* separately that f and g are given by their Taylor series.

(2 marks) (e) $\sin(x) + \cos(x)\exp(-1/x^2)$ (where, by convention we treat $\exp(-1/x^2)$ as 0 at 0).

Solution: Since $\cos(x)$ is bounded for all x, one sees easily that $\cos(x)\exp(-1/x^2)$ has Taylor series with *all* terms 0. It follows that the Taylor series of $\sin(x) + \cos(x)\exp(-1/x^2)$ is the *same* as the Taylor series of $\sin(x)$. In particular, the remainder term cannot be made arbitrarily small for any non-zero value of x.