Logarithm and Exponential

One of the important functions which is *primarily* defined in terms of integration is

$$\log(x) = \int_1^x \frac{dt}{t}$$

This is called the *logarithm* function. Since $x \mapsto 1/x$ is a continuous function on *any* interval $[a, b]$ with $0 < a < b$, the function is well-defined for all positive values of x. (For $x < 1$, it is $-\int_x^1 (dt/t)$ by the usual rules of integration.)

Since $1/x$ is a positive function of x for positive values of x, the integral $\log(x)$ is an increasing function of x; in fact it is strictly monotonic. Moreover, we have

$$\int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t}$$

By the substitution $t = ux$, and the usual rules of integration we have

$$\int_x^{xy} \frac{dt}{t} = \int_1^y \frac{du}{u}$$

In other words, we have $\log(xy) = \log(x) + \log(y)$. Since $\log(1) = 0$ we see that

$$\log : \mathbb{R}_{>0} \to \mathbb{R}$$

is a *group homomorphism* from the multiplicative group of positive real numbers to the additive group of all real numbers; in other words, it turns multiplication into addition. This is the first important use of the logarithm function since multiplication is (on the face of it) much more complicated than addition.

Since log is a strictly monotonic continuous function, it has a continuous inverse. This inverse is also useful since finding the logarithm of the product of two numbers is not the same as finding the product!

As it turns out we have *already* found this inverse and it is exp:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We have already seen that this series converges absolutely and uniformly for x such that $|x| \leq r$ for *all* values of r. It follows that the derivative exists at all points x and is given by term-by-term differentiation as

$$\frac{d}{dx} \exp(x) = \sum_{k=0}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$$

We have also shown that $\exp(x + y) = \exp(x)\exp(y)$ for all x and y. Since the above series consists of positive terms for $x > 0$, it follows that $\exp(x) > 0$

for $x > 0$. Since $\exp(x)\exp(-x) = \exp(x-x) = \exp(0) = 1$ it follows that $\exp(-x) > 0$ for $x > 0$. Thus,

$$\exp : \mathbb{R} \to \mathbb{R}^{>0}$$

is a group homomorphism from the additive group of real numbers to the multiplicative group of positive real numbers.

By the chain rule for differentiation

$$\frac{d}{dx}\log(\exp(x)) = \frac{1}{\exp(x)}\exp(x) = 1$$

Moreover, $\log(\exp(0)) = \log(1) = 0$. Thus, by the fundamental theorem of calculus

$$\log(\exp(x)) = \int_0^x 1 \cdot dt = x$$

This shows that exp and log are inverses of each other!

Powers

Using exp and log it is possible to define the functions $x^r = \exp(r\log(x))$ which make sense for any positive x and any real number r. We can also extend this notion to $x = 0$ for convenience by defining $0^r = 0$ for $r > 0$, $0^0 = 1$ and $0^r = \infty$ for $r < 0$. This convention has been used in the little-o notation as it makes expressions much shorter.

Growth

Just as exp represents growth that is faster than *any* polynomial growth, log represents growth that is *extremely* slow. The sequence $(\log(n))_{n\geq 1}$ *does* grow arbitrarily large as n increases, however, it is slower than $(n^r)_{n\geq 1}$ for *any* positive number r.

Euler-Mascheroni Constant

As seen earlier, $\int_1^x dt/t$ represents the area under the graph $y = 1/x$ over the segment $[1, x]$ of the real line. We note that $1/(n+1) \leq 1/t \leq 1/n$ for t lying in the interval $[n, n+1]$. It follows easily that

$$\sum_{n=1}^{N} \frac{1}{n+1} < \log(N+1) < \sum_{n=1}^{N} \frac{1}{n}$$

Let $H(N) = \sum_{n=1}^{N}(1/n)$ be the partial sum of the Harmonic series. Then this inequality becomes

$$H(N+1) - 1 < \log(N+1) < H(N)$$

We define $a_N = H(N) - \log(N)$ and note that this is an increasing sequence of positive numbers.

Subtracting $\log(N)$ from the first inequality we get

$$H(N+1) - log(N+1) < \log(N+1) + 1 - \log(N)$$

By the multiplicative property of log, we have $\log(N+1) - \log(N) = \log(1 + (1/N))$. By the continuity of log, the sequence $(\log(1+1/N))_{N\geq 1}$ has limit $\log(1) = 0$. In particular, $(a_n)_{n\geq 1}$ is a bounded sequence.

The Euler-Mascheroni constant γ is defined as the least upper bound of $(a_n)_{n\geq 1}$.

Even though this is quite a direct definition, not much is known about this number. In particular, it is not known that it is a rational number (fraction). Some number-theoretic results can be used to show that if it *is* a fraction, then the denominator has at least a million digits! Some people have tried to (jokingly!) argue that this means that, if God exists, it cannot be a rational number.