Power Series (Continued)

We continue our study of power series $\sum_{k=0}^{\infty} a_k x^k$ and some special examples.

We have seen earlier that if, for some positive R, there is a positive constant M and an integer n so that for all $k \geq n$ we have

$$|a_k| < \frac{M}{R^k}$$

then, the series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly and absolutely for the region $|x| \leq r$ for any r such that $0 < r < R$.

We note that we have the equivalent inequality $|a_k|^{1/k} < \frac{M^{1/k}}{R}$ for $k \geq n$. Now, for any positive constant M we can see that $(M^{1/k})_{k \geq n}$ converges to 1. It follows that $\limsup(|a_k|^{1/k})_{k \geq 1} \leq (1/R)$.

Conversely, if $\limsup(|a_k|^{1/k})_{k \geq 1} \leq 1/S$ for some positive constant S. Then, for any R such that $0 < R < S$, we have $1/S < 1/R$. So, by the definition of limit superior, there is an n such that $|a_k|^{1/k} < 1/R$ for all $k \geq n$. It follows that $|a_k| < 1/R^k$ and thus we have uniform and absolute convergence in the region $|x| \leq r$ for any r such that $0 < r < R$. Since this is true for *every* R such that $0 < R < S$, we see that the power series converges uniformly and absolutely in the region $|x| \leq r$ for any r such that $0 < r < S$.

For the reasons above, we put $\limsup(|a_k|^{1/k})_{k \geq 1} = 1/R$ and call R the "radius of convergence". Note that if this limit superior is ∞ (which it *can* be) then $R = 0$. In all other cases $1/R < \infty$, so that $R > 0$. In that case the power series converges uniformly and absolutely in the region $|x| \leq r$ where r is such that $0 < r < R$. **Warning**: Note that the series *does not* in general converge for $|x| = R$.

Error estimates

Even when the power series converges absolutely and uniformly in some region like $|x| \leq r$, it is not always easy to calculate the sum. However, this *does* mean that the series $\sum_{k=0}^{\infty} |a_k| r^k$ converges. As the latter is a series of non-negative terms, the sequence of partial sums is *increasing* (actually it is non-decreasing as some terms of the series may be 0) to its limit. Hence, given any positive integer p, there is a positive integer q such that the $\sum_{k=q+1}^{\infty} |a_k| r^k < 1/p$. Now, for $|x| \leq r$ we get

$$\left| \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{q} a_k x^k \right| \leq \sum_{k=q+1}^{\infty} |a_k| r^k < 1/p$$

Thus, given any positive integer p, we can choose a positive integer q so that the sum $\sum_{k=0}^{q} a_k x^k$ is at most $1/p$ away from the *actual* value of the power series.

Warning: We cannot decide in advance to take only a fixed number of terms.

Warning: It is *not* enough to stop at the point q when $|a_{q+1}x^{q+1}|$ is small enough. The error term is the sum of this term *and* all the remaining terms.

Products of power series

Give power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ which converge uniformly and absolutely in the region $|x| \leq r$ for some positive r. We want to write a power series expression for the product. First of all, we note that *formally* (without worrying about convergence)

$$\left(\sum_{k=0}^{\infty} a_k x^k\right) \cdot \left(\sum_{m=0}^{\infty} b_m x^m\right) = \left(\sum_{n=0}^{\infty} c_n x^n\right)$$

where, by equating coefficients, we obtain

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Let $A = \sum_{k=0}^{\infty} |a_k| r^k$ and $B = \sum_{k=0}^{\infty} |b_k| r^k$. Given a positive integer p. By the convergence of the given power series we can find positive integers u and v so that

$$\sum_{k=u+1} |a_k| r^k < 1/(3pB) \text{ and } \sum_{k=w+1} |b_k| r^k < 1/(3pA)$$

We take $w \geq 2\max\{u, v\}$, then if $k + m > w$, either $k > u$ or $m > v$. Hence, we see that

$$\left|\sum_{k=w+1}^{\infty} c_k x^k\right| \leq \left|\sum_{k=0}^{u} a_k x^k\right| \cdot \left|\sum_{m=v+1}^{\infty} b_m x^m\right| + \left|\sum_{k=u+1}^{\infty} a_k x^k\right| \cdot \left|\sum_{m=0}^{v} b_m x^m\right| + \left|\sum_{k=u+1}^{\infty} a_k x^k\right| \cdot \left|\sum_{m=v+1}^{\infty} b_m x^m\right|$$

Now, we have the inequalities

$$\begin{aligned}
\left|\sum_{k=0}^{u} a_k x^k\right| \cdot \left|\sum_{m=v+1}^{\infty} b_m x^m\right| &\leq \sum_{k=0}^{u} \left|a_k x^k\right| \cdot \sum_{m=v+1}^{\infty} |b_m x^m| &&\leq A\frac{1}{3pA} \\
\left|\sum_{k=u+1}^{\infty} a_k x^k\right| \cdot \left|\sum_{m=0}^{v} b_m x^m\right| &\leq \sum_{k=u+1}^{\infty} \left|a_k x^k\right| \cdot \sum_{m=0}^{v} |b_m x^m| &&\leq \frac{1}{3pB}B \\
\left|\sum_{k=u+1}^{\infty} a_k x^k\right| \cdot \left|\sum_{m=v+1}^{\infty} b_m x^m\right| &\leq \sum_{k=u+1}^{\infty} \left|a_k x^k\right| \cdot \sum_{m=v+1}^{\infty} |b_m x^m| &&\leq \frac{1}{3pB}\frac{1}{3pA}
\end{aligned}$$

We conclude

$$\left|\sum_{k=w+1}^{\infty} |c_k| r^k\right| \leq 1/p$$

It follows that the above *formal* product converges uniformly and absolutely and that it converges to the product of the series.

Examples

We now apply the above calculations to work out some important examples of such products.

Addition law for exp

We have seen that the power series $\sum_{k=0}^{\infty} x^k/k!$ converges absolutely and uniformly in $|x| \leq r$ for *every* positive r. Putting $x = at$ for some constant a, we see that the power series $\sum_{k=0}^{\infty}(a^k/k!)t^k$ converges absolutely and uniformly in $|t| \leq r$ for every positive r. Similarly, for some constant b, the power series $\sum_{k=0}^{\infty}(b^k/k!)t^k$ converges absolutely and uniformly in $|t| \leq r$ for every positive r. We now calculate the product of these series.

$$\left(\sum_{k=0}^{\infty} \frac{a^k}{k!} t^k\right) \cdot \left(\sum_{m=0}^{\infty} \frac{b^m}{m!} t^m\right) = \sum_{n=0}^{\infty} c_n t^n$$

As seen above, we have

$$c_n = \sum_{k=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!}$$

The Binomial theorem gives us

$$\sum_{k=0}^{n} n! \cdot \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} = (a+b)^n$$

It follows that $c_n = (a+b)^n/(n!)$. Using the standard notation $\exp(x)$ for the sum of the power series $\sum_{k=0}^{\infty} x^k/k!$, the above identity becomes

$$\exp(at) \cdot \exp(bt) = \exp((a+b)t)$$

Putting $t = 1$ we obtain $\exp(a)\exp(b) = \exp(a+b)$. In other words, exp turns addition into multiplication.

The power series expression for $\exp(x) = \sum_{k=0}^{\infty} x^k/k!$ is a sum of positive terms when $x > 0$. Hence, it follows that $\exp(x) > 0$ for $x > 0$. Now, using the identity $\exp(x)\exp(-x) = 1$, it follows that $\exp(-x) > 0$ for $x > 0$. This is not obvious from the power series! Since $\exp(0) = 1$, we see that $\exp(x) > 0$ for *all* real numbers x. In other words,

$$\exp : \mathbb{R} \to \mathbb{R}^{>0}$$

By what has been said above this is a *group* homomorphism from the additive group of real numbers to the multiplicative group of positive real numbers.

Addition Law for Sine and Cosine

Recall the power series

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$
$$\cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$$

As above, we will calculate the product $\sin(at)\cos(bt)$ as a power series in t. Since $\sin(at)$ has only odd powers of t and $\cos(bt)$ has only even powers of t, the product is a power series of the form $\sum_{n=0}^{\infty} c_n t^{2n+1}$. As above the coefficient c_n is given by

$$c_n = \sum_{k=0}^{n}\left((-1)^k \frac{a^{2k+1}}{(2k+1)!}\right)\cdot\left((-1)^{n-k}\frac{b^{2(n-k)}}{(2(n-k))!}\right)$$

As before, we note the Binomial theorem

$$\sum_{k=0}^{n}(2n+1)!\left(\left(\frac{a^{2k+1}}{(2k+1)!}\right)\cdot\left(\frac{b^{2(n-k)}}{(2(n-k))!}\right)+\left(\frac{a^{2k}}{(2k)!}\right)\cdot\left(\frac{b^{2(n-k)+1}}{(2(n-k)+1)!}\right)\right) = (a+b)^{2n+1}$$

We note that $(-1)^k\cdot(-1)^{n-k} = (-1)^n$ and that we have only *half* the terms of the above Binomial expansion! To complete it we need $\cos(at)\sin(bt)$. In other words, we see that if $\sum_{n=0} d_n t^{2n+1}$ is the power series of $\sin(at)\cos(bt)+\cos(at)\sin(bt)$, then

$$d^n = (-1)^n\sum_{k=0}^{n}\left(\left(\frac{a^{2k+1}}{(2k+1)!}\right)\cdot\left(\frac{b^{2(n-k)}}{(2(n-k))!}\right)+\left(\frac{a^{2k}}{(2k)!}\right)\cdot\left(\frac{b^{2(n-k)+1}}{(2(n-k)+1)!}\right)\right) = (-1)^n\cdot\frac{(a+b)^{2n+1}}{(2n+1)!}$$

Comparing this with the power series for $\sin((a+b)t)$ we get the identity

$$\sin(at)\cos(bt)+\cos(at)\sin(bt) = \sin((a+b)t)$$

By putting $t = 1$ we get the addition law for Sine

$$\sin(a)\cos(b)+\cos(a)\sin(b) = \sin(a+b)$$

A similar calculation leads us to the identity

$$\cos(a)\cos(b)-\sin(a)\sin(b) = \cos(a+b)$$

In particular, by taking $b = -a$, and using $\sin(-a) = -\sin(a)$ (since the series has only odd powers), we obtain

$$\cos(a)^2+\sin(a)^2 = \cos(0) = 1$$

This shows that $\sin(a)$ and $\cos(a)$ are bounded which is not at all obvious from the power series expression! In particular, we see that the matrix

$$R(a) = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix}$$

has determinant 1. Moreover, the above identities can be written in a concise form as

$$R(a) \cdot R(b) = R(a+b)$$

Since $\cos(0) = 1$ and $\sin(0) = 0$, we also have $R(0)$ is the identity matrix. In other words the map

$$R : \mathbb{R} \to \mathrm{SL}_2(\mathbb{R})$$

is a group homomorphism from the additive group of real numbers to the multiplicative group $\mathrm{SL}_2(\mathbb{R})$ of 2×2 real matrices of determinant 1. In fact, since $R(a)^t = R(-a)$, we see that it lands in the group $\mathrm{SO}_2(\mathbb{R})$ of 2×2 real orthogonal matrices of determinant 1.

The numbers e and π

The number e is defined as $\exp(1)$, we will see later that $\exp(x)$ can be thought of as e^x in some sense.

Defining π takes some more work. In what follows we define:

$\pi = 2\tau$ where τ is the *smallest* zero of $\cos(x)$ in the positive real line.

Note that $\cos(0) = 1$, hence *if* $\cos(x) = 0$ has a solution then (by continuity of cos) there is a smallest positive solution. We will demonstrate the existence of such a solution below.

We compute the series for $\cos(x)$ and $\sin(x)$ for x in the region $[0, 2]$. We have

$$\begin{aligned}
\frac{x^{2k+2}}{(2k+2)!} &= \frac{x^{2k}}{(2k)!} \cdot \frac{x^2}{(2k+1)(2k+2)} &\leq \frac{x^{2k}}{(2k)!} \frac{4}{3 \cdot 4} = \frac{x^{2k}}{(2k)!} \cdot \frac{1}{3} \quad &\text{for } k \geq 1 \\
\frac{x^{2k+3}}{(2k+3)!} &= \frac{x^{2k+1}}{(2k+1)!} \cdot \frac{x^2}{(2k+2)(2k+3)} &\leq \frac{x^{2k}}{(2k)!} \frac{4}{2 \cdot 3} = \frac{x^{2k}}{(2k)!} \cdot \frac{2}{3} \quad &\text{for } k \geq 0
\end{aligned}$$

This means that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \text{ and } \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Are alternating series with *decreasing* terms for x in $[0, 2]$. Thus, the error in the partial sum is *at most* the size of the first term which is dropped *with sign*. Thus,

$$\cos(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} + \sum_{k=3}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!} \leq 1 - 2 + \frac{16}{24} = -\frac{1}{3}$$

In particular, $\cos(2) < 0$. Similarly

$$\sin(x) = x - \frac{x^3}{3!} + \sum_{k=2}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \geq x - \frac{x^3}{6} = \frac{x}{6}(6 - x^2) \geq 0$$

for x in $[0, 2]$.

Now, $\cos(0) = 1$ and $\cos(2) < 0$. By the intermediate value theorem, we see that there is a number τ in the interval $[0, 2]$ for which $\cos(\tau) = 0$. For later use, we note that for this τ we have $\sin(\tau) \geq 0$; since $\cos(\tau)^2 + \sin(\tau^2) = 1$, we get $\sin(\tau) = 1$.

We have seen (using assignment 10 and the fundamental theorem of calculus) that $\cos(x) = 1 - \int_0^x \sin(t)dt$. This shows that $\cos(x)$ is a *decreasing* function of x for x in the range $[0, 2]$. Hence, there is a *unique* solution for $\cos(x) = 0$ in the range $[0, 2]$ and that is τ.

In summary, we have found a number τ (between 0 and 2) such that $\cos(\tau) = 0$, and $\cos(x) > 0$ for $0 \leq x < \tau$. This proves that τ is the smallest positive solution of $\cos(x) = 0$. As mentioned above, we define $\pi = 2\tau$.

We note that $(\cos(\tau), \sin(\tau)) = (0, 1)$ so

$$R(\tau) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hence, $R(\pi) = R(2\tau) = R(\tau)^2 = -I$, where I is the 2×2 identity matrix. Similarly, $R(2\pi) = I$. By the fact that we have a group homomorphism, $R(x + 2\pi) = R(x) \cdot I = R(x)$. Hence, we obtain

$$(\cos(x + 2\pi), \sin(x + 2\pi)) = (\cos(x), \sin(x)$$

In other words, Sine and Cosine are periodic functions. This is *far* from obvious by looking at the power series!

Complex numbers

Matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI + bJ$$

are closed under multiplication. In fact,

$$(a \cdot I + b \cdot J) \cdot (c \cdot I + d \cdot J) = (ac - bd) \cdot I + (ad + bc) \cdot J$$

In particular, we have $J^2 = -I$. Since

$$(a \cdot I + b \cdot J) + (c \cdot I + d \cdot J) = (a + b) \cdot I + (c + d) \cdot J$$

We further note that $\det(a \cdot I + b \cdot J) = a^2 + b^2$ is non-zero unless $a = b = 0$. Hence, the non-zero matrices of this type are invertible.

This collection $\mathbb{C}$ of matrices is called the *field of complex numbers*. We note that if we look at the above multiplication and addition rules *restricted* to matrices with $b = 0$, then we just have the usual laws of arithmetic of numbers except that we are multiplying all numbers by the identity matrix I. Hence, this number (complex) system $\mathbb{C}$ *extends* the usual (real) number system $\mathbb{R}$.

Moreover, we note that each matrix $a \cdot I + b \cdot J$ is *uniquely* determined by the vector

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a \cdot I + b \cdot J) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, we can also identity these matrices with points (a, b) in the plane where multiplication and addition is given by

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

This is called the *Complex Plane* and also denoted as $\mathbb{C}$ under the above identification. As above, the usual real numbers are identified with points of the form $(a, 0)$ which constitute the *Real Line* $\mathbb{R}$ inside the complex plane $\mathbb{C}$. The special number $\iota = (0, 1)$ has the property that $\iota \cdot \iota = (-1, 0)$ which has been identified with the real number 1 as above. Thus, the complex number (a, b) is also written in the form $a + b\iota$ where a and b are real numbers.

Given a complex number $a + b\iota$ and a positive integer n we have *polynomial functions* $P_n(a, b)$ and $Q_n(a, b)$ such that

$$(a + b\iota)^n = P_n(a, b) + Q_n(a, b)\iota$$

We put \$P_0=1 \$ and $Q_0 = 0$ and consider the power series

$$f(t) = \sum_{n=0} \frac{P_n(a, b)}{n!} t^n \text{ and } g(t) = \sum_{n=0} \frac{Q_n(a, b)}{n!} t^n$$

Since $(a + b\iota) \cdot (a - b\iota) = a^2 + b^2$ we can check that

$$P_n(a, b)^2 + Q_n(a, b)^2 = (a^2 + b^2)^n$$

In particular $|P_n(a, b)| \leq (a^2 + b^2)^{n/2}$. Now $\exp((a^2 + b^2)^{1/2} t)$ converges for all values of t. We can use this to show that $f(t)$ and $g(t)$ converge for all values of t. From the identity

$$f(t) + g(t)\iota = \sum_{n=0} \frac{(a + b\iota)^n}{n!} t^n$$

we see that we can think of $f(t) + g(t)\iota$ as $\exp((a + b\iota)t)$.

This motivates us to calculate

$$\begin{aligned}
P_n(a,0) &= a^n \\
Q_n(a,0) &= 0 \\
P_n(0,b) &= \begin{cases} 0 & n = 2k+1 \\ (-1)^k b^{2k} & n = 2k \end{cases} \\
Q_n(0,b) &= \begin{cases} 0 & n = 2k \\ (-1)^k b^{2k+1} & n = 2k+1 \end{cases}
\end{aligned}$$

We then check the identity

$$\exp((b\iota)t) = \cos(bt) + \sin(bt)\iota$$

Note that the latter is identified with the matrix $R(bt)$. We can apply the Binomial Theorem (which has to be re-checked in this new context) to get

$$P_n(a,b) + Q_n(a,b)\iota = (a + b\iota)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \iota^{n-k}$$

A slightly(!) more complicated application of the product rule for power series then allows us to show that:

$$\exp(at)\exp((b\iota)t) = \exp((a + b\iota)t)$$

Putting $t = 1$ this gives

$$\exp(a + b\iota) = \exp(a)\exp(b\iota)$$

Some more algebra then shows us that *this* exp map is also a group homomorphism

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

where the first is considered as an additive group and the latter as the multiplicative group of non-zero complex numbers.

Euler's identity

Using the definition of π given above and the results of the previous section we see that

$$\exp(\pi\iota) = -1$$

This famous identity was first asserted by Euler who then concluded (since it looked miraculous to him) that it proved the existence of an supernatural being!