Taylor Series

Note that a function f is continuous at x_0 if $f = a+g$ where a is a constant and g is $o(|x-x_0|^0)$ at x_0, and a function is differentiable at f at x_0 if $f = a+b(x-x_0)+h$ where a and b are constants and h is $o(|x-x_0|^1)$ are x_0. It thus seems worthwhile to consider the possibility that

$$f = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + f_n$$

where a_i are constants and f_n is $o(|x - x_0|^n)$. Note that this can also be written as

$$f = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + h_n \cdot (x - x_0)^n$$

where h_n is $o(|x - x_0|^0)$; in other words $h_n(x_0) = 0$ and h_n is continuous at x_0.

In the second form, it is reasonably clear that f is differentiable at some $x \neq x_0$ if and only if h_n is differentiable at that x since for such x

$$h_n = \frac{f - (a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n)}{(x - x_0)^n}$$

In particular, this condition *does not necessarily* mean that f is differentiable at any point other than x_0.

Hence, we *cannot* write an expression like

$$f'(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n+1} + (f_n)'$$

since neither f' nor $(f_n)'$ may be defined!

Indefinite integrals

Reversing the process, let us consider an indefinite integral

$$F(x) = b + \int_{x_0}^{x} f(t)dt$$

If we *assume* that f_n is continuous in some region $|x - x_0| < r$ for small enough r, then this integral becomes

$$F(x) = b + a_0(x - x_0) + a_1\frac{(x - x_0)^2}{2} + \cdots + a_n\frac{(x - x_0)^{n+1}}{n + 1} + \int_{x_0}^{x} f_n(t)dt$$

Now, f_n is $o(|x - x_0|^n)$ means that for all k positive integers, there is a positive integer p so that $|f(x)| \leq (1/k)|x - x_0|^n$ for all x in the region $|x - x_0| \leq 1/n$. For such x, we then get

$$\left|\int_{x_0}^{x} f_n(t)dt\right| \leq \left|\int_{x_0}^{x} (1/k)|(t - t_0)^n|dt\right|$$

We calculate

$$\int_0^x |t|^n dt = \begin{cases} \frac{t^{n+1}}{n+1} & t \geq 0 \\ -\frac{(-t)^{n+1}}{n+1} & t \leq 0 \end{cases} = x\frac{|x|^n}{n+1}$$

Applying this, we get

$$\left|\int_{x_0}^x f_n(t)dt\right| \leq (1/k)\frac{|(x-x_0)|^{n+1}}{n+1}$$

It follows easily that if $F_n = \int_{x_0}^x f(t)dt$, the F_n is $o(|x-x_0|^{n+1})$.

In summary, if $f = \sum_{k=0}^n a_k x^k + f_n$ with f_n in $o(|x-x_0|^n)$ at x_0, *and* f_n is continuous at x_0, then, if F is an indefinite integral of f, then

$$F(x) = b + a_0(x-x_0) + a_1\frac{(x-x_0)^2}{2} + \cdots + a_n\frac{(x-x_0)^{n+1}}{n+1} + F_n$$

where F_n is $o(|x-x_0|^{n+1})$ at x_0.

We can apply this to the case where we assume that f is differentiable at all points close enough to x_0 and the derivative f' is continuous for such x. In that case, we have an expression

$$f'(x) = f'(x_0) + f''(x_0)(x-x_0) + g(x)$$

where $g(x)$ is $o(|x-x_0|^1)$ at x_0 *and* g is continuous in a small enough region around x (since it is the difference of two continuous functions). By the fundamental theorem of calculus, f is an indefinite integral of f' and the above calculation says that

$$f(x) = f(x_0) + f'(x_0) + f''(x_0)(x-x_0)/2 + h(x)$$

where $h(x)$ is $o(|x-x_0|^2)$ at x_0.

Multiple differentiability

We define a function f to be 0-times (continuously) differentiable at x_0, if f is continuous at all x in some region $|x-x_0| < r$. Note that this is *stronger* than the hypothesis that it is continuous at x_0.

We now extend this notion to n-times (continuously) differentiable at x_0 for $n > 0$ as follows.

We say that a function f is n-times (continuously) differentiable at x_0, if its derivative $f'(x)$ exists for all x in some region $|x-x_0| < r$ *and* the function f' is $n-1$-times is (continuously) differentiable. We use the notion $f^{(n)} = (f^{(n-1)})'$ is the iterated derivative of f; note the convention that $f^{(0)} = f$ and thus $f^{(1)} = f'$.

Again note that asking for a function to be 1-times (continuously) differentiable is a bit *stronger* than the requirement that $f - l$ is $o(|x-x_0|^1)$ at x_0 for a suitable linear function as we have the requirement that f' exists and is continuous in some region $|x-x_0| < r$.

Extending the arguments given in the previous section we can show that if f is n-times differentiable at x_0 in this sense, then we have the expression (n-truncated Taylor series of f)

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + h$$

where h is $o(|x - x_0|^n)$ at x_0. Note that the given condition on f means that h is *also* n-times differentiable in some region $|x - x_0| < r$.

By induction, we have an expression

$$f'(x) = f'(x_0) + f''(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + g$$

where g is $o(|x - x_0|^n)$. Since f' is $(n-1)$-times differentiable, so is g. In particular, it is continuous and so the results of the previous section apply. Thus, with $h = \int_{x_0}^{x} g(t)dt$ we obtain the required expression.

Examples

We look at some examples of the above calculations.

Power Series

Suppose that f is given by a power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ which converges uniformly and absolutely in some region $|x - x_0| \le r$. As seen in Assignment 10, this means that for any chosen non-negative integer n, the series $\sum_{k=n+1}^{\infty} a_k x^{k-n+1}$ converges absolutely and uniformly to a function g in the same region. Moreover, we have also seen in Assignment 10 that both f and g are differentiable any number of times. We easily obtain the expression

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + g(x)(x - x_0)^{n+1}$$

Since g is continuous at x_0, we see that $g(x)(x - x_0)^{n+1}$ is $o(|x - x_0|^n)$ and so the above is the n-truncated Taylor expansion of f at x_0. We thus obtain the identities

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

which we can also obtain by calculating the derivatives term-by-term as in Assignment 10.

Given a function f which is n-times differentiable at x_0 for *every* n, this *may* tempt us into thinking that f is given by the power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ with $a_k = f^{(k)}/(k!)$. However, as we shall see below, this is *not true*!

Powers of $|x|$

Recall that $|x|$ is defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

We note that the derivative does not exist at $x = 0$. However, the function *is* continuous and $o(|x|^0)$ at 0. Consider the functions

$$|x|^n = \begin{cases} x^n & x \geq 0 \\ (-x)^n & x \leq 0 \end{cases}$$

We note that, if $n > 1$, then its derivative at $x = 0$ exists. In fact, for $x \neq 0$, we can apply the Chain Rule to get

$$\frac{d}{dx}|x|^n = \begin{cases} nx^{n-1} & x > 0 \\ (-1)^n nx^{n-1} & x < 0 \end{cases}$$

Since $n > 1$ we see that this function has a continuous extension to 0 given by $n|x|x^{n-1}$, which has the value 0 at 0. It follows that the n-truncated Taylor series of $|x|^n$ is

$$|x|^n = 0 + |x|^n$$

In other words, the n-truncated Taylor series only has the "remainder term" $g(x) = |x|^n$. The same argument can be made for $|x|^a x^b$ for any positive integers a and b! So the "terms" of the Taylor series do not give us any information about these functions other than that they are $o(|x|^{a+b-1}$ at 0.

More pathologies

Let us examine the function below

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Fix a positive integer n. We first claim that $f(x)$ is $o(|x|^{2n})$ at 0.

We know that $\exp(y)$ grows faster than y^n. In fact, $\exp(y) > y^{n+1}/(n+1)!$ and the latter grows faster than y^n. Given a positive integer k, if $y > (n+1)!k$, then $\exp(y) > ky^n$. Now, put $y = 1/x^2$, the condition $y > (n+1)!k$, becomes $x^2 < 1/((n+1)!k)$. Choose a p so that $p^2 > (n+1)!k$. Thus, we see that if $|x| < 1/p$, then $\exp(1/x^2) > k1/x^{2n}$. Dividing both sides by $\exp(1/x^2)$ and multiplying by $(1/k)|x|^{2n}$, we get $(1/k)|x|^{2n} > |\exp(-1/x^2)|$. In other words, given a positive integer k, we have found a p so that for x in the region $0 < |x| < 1/p$ we have $|\exp(-1/x^2)| \leq (1/k)|x|^{2n}$. This condition is obvious for $x = 0$. Thus we have proved the claim.

Now, for *any* continuous function g and for any integer n it follows from what we have proved about asymptotic behaviour that the function

$$h(x) = \begin{cases} \frac{g(x)}{x^n} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is in $o(|x|^m$ for any positive m. In particular, it is continuous at 0 and vanishes there. By the standard arithmetic properties of continuous functions it is also continuous for $x \neq 0$.

Now the function

$$h_1(x) = \begin{cases} \frac{h(x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is also continuous at 0 and vanishes there for the same reason. As seen earlier, this means that the function h is differentiable at 0; moreover $(dh/dx)(0) = h_1(0) = 0$.

Now assume that g is differentiable. The derivative of $(g(x)/x^n)\exp(-1/x^2)$ at $x \neq 0$ can be found by the application of the rules of differentiation:

$$\frac{d}{dx}\left(\frac{g(x)}{x^n}\exp(-1/x^2)\right) = \left(\frac{g'(x)}{x^n} - n\frac{g(x)}{x^{n+1}} - \frac{g(x)}{x^n}\cdot\frac{2}{x^3}\right)\exp(-1/x^2) = \frac{g_1(x)}{x^{n+4}}\exp(-1/x^2)$$

where $g_2(x) = x^3 g'(x) - nx^3 g(x) - xg(x)$. We now put

$$h_2(x) = \begin{cases} \frac{g_2(x)}{x^{n+4}} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then, by the calculation above we see that $h_2(x)$ is the derivative of h at x for *all* x. Moreover, as seen above h_2 is continuous at 0. In other words, we have shown that h is continuously differentiable and its derivative at 0 “of the same type”.

Note that if g is a polynomial function then so is g_2. Thus, if we start with the original function $f(x)$, we see that *all* its derivatives are of type similar to h with g a polynomial. (One proves this by induction.) All of these vanish at 0.

We conclude that f is a n-times differentiable function *all* of whose derivatives at 0 are 0. However, it is obvious that $f(1) = e^{-1} \neq 0$. Thus, we have a non-zero function whose Taylor series is 0.