Solutions to Assignment 10

1. Comparison of radius of convergence of various power series.

(1 mark) (a) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely and uniformly for $|x| \leq r$ for some $r > 0$, show that $\sum_{k=1}^{\infty} a_k x^{k-1}$ also converges absolutely and uniformly for the same values of x.

> **Solution:** The given convergence condition says that: Given any positive integer p, there is a positive integer n so that $\sum_{k=n}^{m} |a_k| r^k < 1/p$ for all $m \geq n$. It follows that $\sum_{k=n}^{m} |a_k| r^{k-1} < 1/(pr)$.
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> Now, given any positive integer q we choose (by the Archimedean principle) a positive integer p such that $pr > q$. We then choose n as above for this p. It follows that $\sum_{k=n}^{m} |a_k| r^{k-1} < 1/q$. Since this can be done for every positive integer q, it follows that $\sum_{k=1}^{\infty} |a_k x^{k-1}$ is absolutely and uniformly convergent for $|x| \leq r$.

(1 mark) (b) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely and uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} a_k x^{k+1}$ also converges absolutely and uniformly for the same values of x.

> **Solution:** The sequence of partial sums $s_n(x) = \sum_{k=0}^{n} a_k x^k$ converges absolutely and uniformly in $|x| \leq r$. Given a positive integer p, choose a positive integer q so that $q > pr$. By uniform convergence of $(s_n)_{n\geq 1}$ there is an n so that $\|s_n - s_m\| < 1/q$ for $m \geq n$. It follows tha easily that $\|xs_n(x) - xs_m\| < r/q < 1/p$.
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> Since this can be done for every positive integer p, it follows that $\sum_{k=0}^{\infty} |a_k x^{k+1}$ is absolutely and uniformly convergent for $|x| \leq r$.

(1 mark) (c) Show that the following numbers are all equal

$$\limsup(|a_n|^{1/n})_{n\geq 0} = \limsup(|a_n|^{1/(n-1)})_{n\geq 1} = \limsup(|a_n|^{1/(n+1)})_{n\geq 0} = \limsup(|a_n|^{1/n})_{n\geq p}$$

(Hint: Use the previous exercises.)

> **Solution:** It has been seen that the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is $R = \limsup(|a_n|^{1/n})_{n\geq 0}$. This means that the series converges absolutely and uniformly for $|x| \leq r$ where $0 < r < R$.
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> By the previous exercises this also gives the convergence of $\sum_{k=1}^{\infty} a_k x^{k-1}$ and $\sum_{k=0}^{\infty} a_k x^{k+1}$. This gives the first two equalities.
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> For the final equality, note that lim sup, like all other limiting operations "does not care" about the first few terms of a sequence. Specifically,
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> $$\limsup(b_n)_{n\geq 1} = \inf(\sup(b_n)_{n\geq k})_{k\geq 1}$$

Since $((\sup(b_n))_{n\geq k})_{k\geq 1}$ is a *decreasing* sequence, we have

$$\inf(\sup(b_n)_{n\geq k})_{k\geq 1} = \inf(\sup(b_n)_{n\geq k})_{k\geq p} = \limsup(b_n)_{n\geq q}$$

(1 mark) (d) Show that

$$\lim(n^{1/n})_{n\geq 1} = 1$$

Solution: Since $x^{1/n}$ is the monotonically increasing inverse of the monotonically increasing function x^n, we see that $n^{1/n} \geq 1$ for $n \geq 1$. So we easily see that $\liminf((n^{1/n})_{n\geq 1} \geq 1$.

Given any $x > 0$, we have seen that the function $(1+x)^n$ grows faster than n. It follows that there is a positive integer m so that $(1+x)^n > n$ for all $n \geq m$. It follows that $1+x > n^{1/n}$ for $n \geq m$. Hence, we see that $\limsup((n^{1/n})_{n\geq 1} \leq 1+x$. Since this is true for all positive x, it follows that $\limsup((n^{1/n}))_{n\geq 1} \leq 1$.

Combining the two inequalities, we get

$$1 \leq \liminf((n^{1/n})_{n\geq 1} \limsup((n^{1/n}))_{n\geq 1} \leq 1$$

Hence, all of the above are equal and we have the desired result.

(1 mark) (e) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1}$ also converges uniformly for the same values of x.

Solution: It follows from part 3 above that

$$\limsup(|a_n|^{1/n})_{n\geq 1} = \limsup(|a_n|^{1/(n+1)})_{n\geq 1}$$

Given $\lim(b_n)_{n\geq 1} = r$ and $\limsup(c_n)_{n\geq 1} = s$, one shows that $\limsup(b_n c_n)_{n\geq 1} = rs$. Hence, using part 4 we have

$$\limsup(|a_n|^{1/(n+1)})_{n\geq 1} = \limsup(|a_n/(n+1)|^{1/(n+1)})_{n\geq 1}$$

The required result follows from the formula for the radius of convergence of power series.

(1 mark) (f) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} k a_k x^{k-1}$ also converges uniformly for the same values of x.

Solution: It follows from part 3 above that

$$\limsup(|a_n|^{1/n})_{n\geq 1} = \limsup(|a_n|^{1/n})_{n\geq 2} = \limsup(|a_n|^{1/(n-1)})_{n\geq 2}$$

Given $\lim(b_n)_{n\geq 1} = r$ and $\limsup(c_n)_{n\geq 1} = s$, one shows that $\limsup(b_n c_n)_{n\geq 1} = rs$. Hence, using part 4 we have

$$\limsup(|a_n|^{1/(n-1)})_{n\geq 2} = \limsup(|na_n|^{1/(n-1)})_{n\geq 2}$$

The required result follows from the formula for the radius of convergence of power series.

2. Use the previous exercise to calculate the derivatives of the following power series *and* recognise the resulting function:

(1 mark) (a) $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)!}$.

(1 mark) (b) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

(1 mark) (c) $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

(1 mark) (d) $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

Solution: By the previous exercise, the power series obtained by term-by-term differentiation of a power series converges uniformly in the same region where the power series converges uniformly.

Since we have checked that the integrals of polynomials are the also given by term-by-term integration, it follows that the integrals of this differentiated power series converge to the original function. By the fundamental theorem of calculus, we see that the derivative of the power series is given by term by term differentiation. We now easily check the formulas

$$\begin{aligned}
\frac{d}{dx}\exp(x) &= \exp(x) \\
\frac{d}{dx}\cos(x) &= -\sin(x) \\
\frac{d}{dx}\sinh(x) &= \cosh(x) \\
\frac{d}{dx}\log(1+x) &= \frac{1}{1+x}
\end{aligned}$$

The first three series converge for all values of x and the last series converges aboslutely and uniformly for $|x| \leq r < 1$ for all r such that $0 < r < 1$.