Asymptotic Behaviour

We started the course by trying to understand how we can compare the growth of various functions of positive integers for sufficiently large values. Now that we are looking at functions of real numbers one can study something similar at any point on the real line.

We say that a function f vanishes to order *greater than* r ($r \geq 0$) at a point x_0 if, given any positive integer k, there is a positive integer n such that

$$|f(x)| \leq (1/k)|x - x_0|^r$$

for *all* x such that $|x - x_0| \leq (1/n)$. (Here and later we follow the convention that $x^0 = 1$ for all $x \geq 0$.)

The above is also written as f is $o(|x - x_0|^r)$ at x_0. This helps us to generalise further (a favourite pastime of mathematicians!) and think of defining $o(g)$ for some function g!

Let us see how this explains some of the notions we have worked with so far.

Continuity

> A function f is continuous at a point x_0 if $f - f(x_0)$ is $o(|x - x_0|^0)$ at x_0.

Recall that f is continuous at x_0, if, given any positive integer k, there is a positive integer n so that $|f(x) - f(x_0)| \leq (1/k)$ for all x such that $|x - x_0| \leq 1/n$. This is precisely the given condition.

Differentiability

> A function f has derivative $f'(x_0)$\$ at a point x_0 if and only if $f - f(x_0) - f'(x_0)(x - x_0)$ is $o(|x - x_0|^1)$ at x_0.

Recall that if f has derivative $f'(x_0)$\$ at a point x_0, then for every positive integer k, there is a positive integer n so that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq (1/k)|x - x_0|$$

for all x such that $|x - x_0| \leq 1/n$. This is precisely the given condition.

Algebraic Properties

We now show a number of algebraic properties of this notion of a function being $o(|x - x_0|^r)$ at x_0.

Multiplication by a constant

> If f is $o(|x - x_0|^r)$ at x_0 and K is any constant, then Kf is $o(|x - x_0|^r)$ at x_0 as well.

Given a positive integer k, by the Archimedean principle, choose a positive integer p so that $p > k|K|$. Now, since f is $o(|x-x_0|^r)$ at x_0, there is a positive integer n so that $|f(x)| \leq (1/p)|x-x_0|^r$. It follows that $|Kf(x)| \leq (1/k)|x-x_0|^r$ for the same range of values of x.

Addition

> If f and g are *both* $o(|x-x_0|^r)$ at x_0, then so is $f+g$.

Given a positive integer k, we have positive integers n and m such that

- $|f(x)| \leq (1/2k)|x-x_0|^r$ for x such that $|x-x_0| < 1/n$
- $|g(x)| \leq (1/2k)|x-x_0|^r$ for x such that $|x-x_0| < 1/m$

If $q = \max\{m, n\}$, then we have

$$|(f+g)(x)| \leq |f(x)| + |g(x)| \leq (1/k)|x-x_0|^r$$

for x such that $|x-x_0| < 1/q$.

Nested conditions

> If f is $o(|x-x_0|^r)$ at x_0 and $r \geq s \geq 0$, then f is also $o(|x-x_0|^s)$ at x_0.

Given a positive integer k we choose a positive integer n so that $|f(x)| \leq (1/k)|x-x_0|^r$ for x such that $|x-x_0| \leq 1/n$. Since n is a positive integer $|x-x_0| \leq 1$ it follows that $|x-x_0|^{r-s} \leq 1$. Hence

$$|f(x)| \leq (1/k)|x-x_0|^{r-s}|x-x_0|^s \leq (1/k)|x-x_0|^s$$

> The function $|x-x_0|^r$ is $o(|x-x_0|^s$ for $r > s \geq 0$.

Since the function n^{r-s} is an increasing function of n, given a positive integer k, there is a positive integer n so that $n^{r-s} > k$. It follows that

$$|x-x_0|^r = |x-x_0|^{r-s}|x-x_0|^s \leq 1/n^{r-s}|x-x_0|^s < (1/k)|x-x_0|^s$$

Product of functions

> Given that f is $o(|x-x_0|^r)$ at x_0 and g is $o(|x-x_0|^s)$ at x_0, the product $f \cdot g$ is $o(|x-x_0|^{r+s})$ at x_0.

Given a positive integer k, we have positive integers n and m such that

- $|f(x)| \leq (1/k)|x-x_0|^r$ for x such that $|x-x_0| < 1/n$
- $|g(x)| \leq (1/k)|x-x_0|^s$ for x such that $|x-x_0| < 1/m$

It follows that

$$|(f \cdot g)(x)| = |f(x)g(x)| \leq (1/k^2)|x-x_0|^{r+s} \leq (1/k)|x-x_0|^{r+s}$$

since $k \geq 1$ means $1/k \leq 1$. Hence we get the required condition.

In particular, we note that if f is $o(|x-x_0|^r)$ and g is a *continuous* function

> Given that f is $o(|x-x_0|^r)$ at x_0 and g is continuous at x_0, the product $f \cdot g$ is also $o(|x-x_0|^r)$ at x_0.

This is *not* a special case of the earlier result since we are not assuming that $g(x_0)=0$.

Division

> Given that f is $o(|x-x_0|^r)$ at x_0 with $r \geq 1$, there is a function g which is $o(|x-x_0|^{r-1})$ at x_0 such that $f(x)=g(x)(x-x_0)$.

Let us define g as follows:

$$g(x) = \begin{cases} 0 & x = x_0 \\ \frac{f(x)}{x-x_0} & x \neq x_0 \end{cases}$$

We will see below that if f is $o(|x-x_0|^r)$ (for any $r \geq 0$), then $f(x_0)=0$. So we do get $f(x)=g(x)(x-x_0)$ for all x. Given a positive integer k, there is a positive integer n so that, for all x with $|x-x_0| \leq 1/n$ we have

$$|f(x)| \leq (1/k)|x-x_0|^r$$

When $x \neq x_0$, this gives

$$|g(x)| \leq (1/k)|x-x_0|^{r-1}$$

On the other hand, since $r \geq 1$, we have

$$|g(x_0)| = 0 \leq (1/k)|x_0-x_0|^{r-1}$$

since our convention is that $x^0 = 1$.

A similar argument can be used to show that:

> Given that f is $o(|x-x_0|^r)$ at x_0 with $r \geq 1$, there is a function g which is $o(|x-x_0|^0)$ at x_0 such that $f(x)=g(x)|x-x_0|^r$.

In this case, we define g as follows:

$$g(x) = \begin{cases} 0 & x = x_0 \\ \frac{f(x)}{|x-x_0|^r} & x \neq x_0 \end{cases}$$

Note that in both cases g is a continuous function vanishing at x_0.

Applications

The above algebraic properties have a few applications.

Value

If f is $o(|x-x_0|^0)$ at x_0 then $f(x_0)=0$. Note that the condition means that $|f(x_0)| \leq (1/k)$ for all positive integers k. Hence the result follows.

Derivative

If $f - g$ is $o(|x - x_0|^1)$ at x_0 and if f is differentiable at x_0, then so is g and the value $g(x_0)$ and the derivative $g'(x_0)$ of g at x_0 are the same as those of f.

As seen above, if $l(x) = f(x_0) + f'(x_0)(x - x_0)$ where $f'(x_0)$ is the derivative of f at x_0, then $f - l$ is $o(|x - x_0|^1)$ at x_0. By the additivity property we see that $g - l = (f - l) - (g - l)$ is also $o(|x - x_0|^1)$ at x_0. It follows that g is differentiable at x_0 and its value and derivative are the same as those of f.

Linearity of Derivative

Using the additivity of the $o()$ conditions, we can easily show that if f and g are differentiable at x_0 and a is some constant, then $af + g$ is differentiable at x_0. Moreover, if $f'(x_0)$ and $g'(x_0)$ denote the derivatives of f and g at x_0, then the derivative of $af + g$ at x_0 is $af'(x_0) + g'(x_0)$. This is left as an exercise.

Product Rule

Given f and g are differentiable at x_0. Let $l(x) = f(x_0) + f'(x_0)(x - x_0)$ and $m(x) = g(x_0) + g'(x_0)(x - x_0)$. We know that $f - l$ and $g - m$ are $o(|x - x_0|^1)$ at x_0. We then have

$$f \cdot g = (l + (f - l)) \cdot (m + (g - m)) = l \cdot m + (f - l) \cdot m + l \cdot (g - m) + (f - l) \cdot (g - m)$$

Since l and m are continuous functions $(f - l) \cdot m$ and $l \cdot (g - m)$ are also $o(|x - x_0|^1)$ at x_0. Now, $(f - l) \cdot (g - m)$ is $o(|x - x_0|^2)$ at x_0, so it is also $o(|x - x_0|^1)$ as seen above.

We conclude that $f \cdot g - l \cdot m$ is $o(|x - x_0|^1)$ at x_0. Thus, one of them is differentiable at x_0 if and only if the other one is; moreover, their values and derivatives are the same. Now

$$\begin{aligned} l(x)m(x) = f(x_0)g(x_0) + f'(x_0)g(x_0)(x - x_0) + f(x_0)g'(x_0)(x - x_0) \\ + f'(x_0)g'(x_0)(x - x_0)^2 \end{aligned}$$

As seen above the function $f'(x_0)g'(x_0)(x - x_0)^2$ is $o(|x - x_0|^1)$ at x_0. So $l \cdot m - L$ is $o(|x - x_0|^1)$ where

$$L(x) = f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0)$$

It follows that the derivative of $f \cdot g$ at x_0 is

$$f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

This rule is called the product rule for differentiation or the Leibniz rule.

Chain Rule

Given a function f which is differentiable at x_0 and takes the value $y_0 = f(x_0)$, and a function g which is differentiable at y_0, we want to calculate the derivative of $g \circ f$ (where $(g \circ f)(x) = g(f(x))$) at x_0.

Let $f'(x_0)$ denote the derivative of f at x_0. As seen above, $f - f(x_0) - f'(x_0)(x - x_0)$ is $o(|x - x_0|^1)$. It further follows that there is a function $f_1(x)$ which is $o(|x - x_0|^0)$ such that

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = f_1(x)(x - x_0)$$

In other words, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f_1(x)(x - x_0)$$

where f_1 is $o(|x - x_0|^0)$. Similarly, we have

$$g(y) = g(y_0) + g'(y_0)(y - y_0) + g_1(y)(y - y_0)$$

where g_1 is $o(|y - y_0|^0)$. We now calculate $g(f(x))$ by substitution (note that $y_0 = f(x_0)$):

$$\begin{aligned} g(f(x)) &= g(f(x_0)) + g'(f(x_0))\,(f(x) - f(x_0)) + g_1(f(x))\,(f(x) - f(x_0)) \\ &= g(f(x_0)) + g'(f(x_0))f'(x_0)(x - x_0) \\ &+ g'(f(x_0))f_1(x)(x - x_0) + g_1(f(x))f'(x_0)(x - x_0) + g_1(f(x))f_1(x)(x - x_0) \end{aligned}$$

We examine the last three terms to show that they vanish to order greater than 1 at x_0.

- First of all $f_1(x)$ is $o(|x - x_0|^0)$ so $g'(f(x_0))f_1(x)(x - x_0)$ is $o(|x - x_0|^1)$ at x_0.
- Secondly, g_1 is continuous at $y_0 = f(x_0)$ and f is continuous at x_0 so $g_1(f(x))$ is continuous at x_0. Thus $g_1(f(x))f_1(x)(x - x_0)$ is $o(|x - x_0|^1)$ at x_0.
- Finally, $g_1(y_0) = 0$ and $f(x_0) = y_0$, thus $g_1(f(x))$ is $o(|x - x_0|^0)$ at x_0 (since it is continuous at has value 0 at x_0). Thus $g_1(f(x))f'(x_0)(x - x_0)$ is $o(|x - x_0|^1)$ at x_0.

In summary, we have shown that

$$g(f(x)) - g(f(x_0)) - g'(f(x_0))f'(x_0)(x - x_0)$$

is $o(|x - x_0|^1)$ at x_0. It follows that $g \circ f$ is differentiable at x_0 and its derivative at x_0 is

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$$

This is called the Chain Rule for differentiation.

Properties of differentiation

There are two notations used for the derivative of a function f at x_0. One is $f'(x_0)$ and the second is $(df/dx)(x_0)$. The first is obvious easier to write but can lead to some ambiguities. The second one is a bit cumbersome but has some notational advantages.

Well-defined: : Given a function f which is differentiable at x_0, there is a uniquely defined number a such that $(df/dx)(x_0) = a$.

Linearity: : Given functions f and g that are differentiable at x_0 and a constant a, the function $af + g$ is differentiable at the point x_0 and

$$\frac{d(af+g)}{dx}(x_0) = a\frac{df}{dx}(x_0) + \frac{dg}{dx}(x_0)$$

Leibniz Rule: : Given functions f and g differentiable at the point x_0, the function $f \cdot g$ is differentiable at the point x_0 and we have

$$\frac{d(f \cdot g)}{dx}(x_0) = \frac{df}{dx}(x_0)g(x_0) + f(x_0)\frac{dg}{dx}(x_0)$$

Chain Rule: : Given a function f differentiable at the point x_0 and a function g differentiable at the point $f(x_0)$, the function $g \circ f$ (defined by $(g \circ f)(x) = g(f(x))$) is differentiable at x_0 and

$$\frac{d(g \circ f)}{dx}(x_0) = \frac{dg}{dx}(f(x_0))\frac{df}{dx}(x_0)$$