Power Series and Derivatives

1. Comparison of radius of convergence of various power series.

(1 mark) (a) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely and uniformly for $|x| \leq r$ for some $r > 0$, show that $\sum_{k=1}^{\infty} a_k x^{k-1}$ also converges absolutely and uniformly for the same values of x.

(1 mark) (b) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely and uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} a_k x^{k+1}$ also converges absolutely and uniformly for the same values of x.

(1 mark) (c) Show that the following numbers are all equal

$$\limsup(|a_n|^{1/n})_{n\geq 0} = \limsup(|a_n|^{1/(n-1)})_{n\geq 1} = \limsup(|a_n|^{1/(n+1)})_{n\geq 0} = \limsup(|a_n|^{1/n})_{n\geq p}$$

(Hint: Use the previous exercises.)

(1 mark) (d) Show that

$$\lim(n^{1/n})_{n\geq 1} = 1$$

(1 mark) (e) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1}$ also converges uniformly for the same values of x.

(1 mark) (f) Given that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $|x| \leq r$, show that $\sum_{k=0}^{\infty} k a_k x^{k-1}$ also converges uniformly for the same values of x.

2. Use the previous exercise to calculate the derivatives of the following power series *and* recognise the resulting function:

(1 mark) (a) $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)!}$.

(1 mark) (b) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

(1 mark) (c) $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

(1 mark) (d) $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.