Mean Values, Indefinite Integrals and Derivatives

In this section we discuss three topics. The relation between these topics will become clear as we go along.

Mean Values

We can think of the integral as "continuous version of sum". With this sense, given a continuous function f on the interval, the number

$$E(f,[a,b]) = \frac{\int_a^b f(x)dx}{b-a}$$

can be thought of as the "mean value" of the function over this interval. When f is non-negative, we know that $\int_a^b f(x)dx = I(f,[a,b])$ is the "area under the curve". Suppose we think of the graph of f over $[a,b]$ as the "surface" of a liquid poured into a (2-dimensional) cup whose base is the interval $[a,b]$ on the x-axis and sides are the vertical lines at a and b. When the liquid "settles down" the top will become horizontal and its height will be $E(f,[a,b])$.

We can ask whether this "mean" value is *in fact* the value of f at some point. To see this consider the function $g = f - E(f,[a,b])$. We note that

$$\int_a^b g(x)dx = \int_a^b f(x)dx - E(f,[a,b]) \int_a^b dx = I(f,[a,b]) - E(f,[a,b])(b-a) = 0$$

Now, if $g > 0$ for all x in the interval $[a,b]$) then $\int_a^b g(x)dx > 0$ as seen earlier. Similarly, if $g < 0$ for all x in the interval $[a,b]$) then $\int_a^b g(x)dx < 0$ as seen earlier. It follows that g take both signs on the interval $[a,b]$. By the intermediate value theorem we see that $g(c) = 0$ for *some* c in the interval $[a.b]$. This means $f(c) = E(f,[a,b])$. So we see that the mean value is the value of f at c.

Indefinite integrals

Given a continuous function f on an interval $[a,b]$ and any x in this interval, we have a function

$$F(x) = \int_a^x f(t)dt = I(f,[a,x])$$

The function F (or more generally the function $F + c$ for a constant c) is called the indefinite integral of f. Sometimes, it is written as $\int f(x)dx$; no specific limits are put on the integral sign to indicate that it is indefinite. It is also a function on $[a,b]$.

Returning to the case of a general continuous function, we note that

$$F(y) - F(x) = \int_x^y f(t)dt$$

In particular, we see that $F(y) - F(x) = E(f, [x, y])(y - x)$. As seen above, there is a point z in the interval $[x, y]$ such that $E(f, [x, y]) = f(z)$. In other words, we have shown that given x and y in the interval $[a, b]$, there is a point z lying between x and y so that

$$F(y) = F(x) + f(z)(y - x)$$

Comparing this with $l(y) = F(x) + f(x)(y - x)$ we see that the difference is $(f(z) - f(x))(y - x)$.

By the continuity of f, given a positive integer k, we can find an n so that $|u - x| < 1/n$ implies that $|f(u) - f(x)| < 1/k$ for all u in the interval $[a, b]$.

Now if $|y - x| < 1/n$, then, since z lies between x and y, we have $|z - x| < 1/n$ and so $|f(z) - f(x)| < 1/k$. It follows that

$$|F(y) - F(x) - f(x)(y - x)| = |f(z) - f(x)||y - x| \leq (1/k)|y - x|$$

We can summarise as follows.

> Given any positive integer k, there is a positive integer n so that if $|y - x| < 1/n$, then $|F(y) - F(x) - f(x)(y - x)| < 1/k|y - x|$.

Another way to say this is that within $1/n$ of x, the function F behaves like a linear function with slope $f(x)$ up to a error at most $(1/k)$ times the distance from x.

Derivative of a function

The above motivates the notion of the derivative of a function. We say that a function F is differentiable at x with derivative a if, for any positive integer k, there is a positive integer n so that

$$|F(y) - F(x) - a(y - x)| < (1/k)|y - x| \text{ for } y \text{ such that } |y - x| < 1/n$$

A different way to say is that the function

$$F_1(y) = \begin{cases} \frac{F(y)-F(x)}{y-x} & y \neq x \\ a & y = x \end{cases}$$

is a continuous function at x.

The statement proved at the end of the previous section is then the statement that an indefinite integral $F(x) = \int_a^x f(t)dt$ is a differentiable function with derivative $f(x)$ at the point x. This is called the *fundamental theorem of calculus*. Loosely speaking, we can say that taking the derivative is the inverse of the process of taking the indefinite integral.

The above proof of the fundamental theorem depends on the continuity of f. However, there are versions of this theorem that can weaken this assumption since there *are* many other functions for which integration makes sense.