Solutions to Assignment 9

1. The following questions are about continuous functions on an interval $[a, b]$ and convergence is with respect to $\|\cdot\|_{[a,b]}$ (in other words, uniform convergence).

(1 mark) (a) Show that

$$\|f + g\| \leq \|f\| + \|g\|$$

Solution: We have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

As seen in the notes there are points c, d, e in $[a, b]$ so that for all x in $[a, b]$ we have

$$\begin{aligned} |f(x) + g(x)| &\leq |f(c) + g(c)| &= \|f + g\| \\ |f(x)| &\leq |f(d)| &= \|f\| \\ |g(x)| &\leq |g(e)| &= \|g\| \end{aligned}$$

It follows that

$$\begin{aligned} \|f + g\| &== |f(c) + g(c)| \\ &\leq |f(c)| + |g(c)| \leq |f(d)| + |g(e)| \\ &= \|f\| + \|g\| \end{aligned}$$

(1 mark) (b) Show that

$$\|f \cdot g\| \leq \|f\| \cdot \|g\|$$

Solution: We have

$$|f(x) \cdot g(x)| = |f(x)| \cdot |g(x)|$$

As seen in the notes there are points c, d, e in $[a, b]$ so that for all x in $[a, b]$ we have

$$\begin{aligned} |f(x) \cdot g(x)| &\leq |f(c) \cdot g(c)| &= \|f \cdot g\| \\ |f(x)| &\leq |f(d)| &= \|f\| \\ |g(x)| &\leq |g(e)| &= \|g\| \end{aligned}$$

It follows that

$$\begin{aligned} \|f \cdot g\| &== |f(c) \cdot g(c)| \\ &= |f(c)| \cdot |g(c)| \leq |f(d)| \cdot |g(e)| \\ &= \|f\| + \|g\| \end{aligned}$$

(1 (bonus)) (c) Find an example of non-zero functions f and g on $[0,1]$ such that $f \cdot g = 0$.

(1 mark) (d) If $(f_n)_{n\geq 1}$ converges to f and $(g_n)_{n\geq 1}$ converges to g, then show that $(f_n + g_n)_{n\geq 1}$ converges to $f + g$.

Solution: Given a positive integer k, choose n_1 so that

$$\|f_n - f\| < 1/(2k) \text{ for all } n \geq n_1$$

and n_2 so that

$$\|g_n - g\| < 1/(2k) \text{ for all } n \geq n_2$$

It follows that, for $n \geq \max\{n_1, n_2\}$ we have

$$\begin{aligned} \|(f_n + g_n) - (f + g)\| &\leq \|f_n - f\| + \|g_n - g\| \\ &< 1/(2k) + 1/(2k) = 1/k \end{aligned}$$

Since we can do this for every positive integer k, the conclusion follows.

(1 mark) (e) If $(f_n)_{n\geq 1}$ converges to f and $(g_n)_{n\geq 1}$ converges to g, then show that $(f_n \cdot g_n)_{n\geq 1}$ converges to $f \cdot g$.

Solution: Choose $M > 1$ so that $\|f\| < M$ and $\|g\| < M$.
Given a positive integer k, choose n_1 so that

$$\|f_n - f\| < 1/(3Mk) \text{ for all } n \geq n_1$$

and n_2 so that

$$\|g_n - g\| < 1/(3Mk) \text{ for all } n \geq n_2$$

It follows that, for $n \geq \max\{n_1, n_2\}$ we have

$$\begin{aligned} \|(f_n \cdot g_n) - (f \cdot g)\| &= \|(f_n - f) \cdot g_n + f \cdot (g_n - g)\| \\ &= \|(f_n - f) \cdot (g_n - g) + (f_n - f) \cdot g + f \cdot (g_n - g)\| \\ &\leq \|(f_n - f) \cdot (g_n - g)\| + \|(f_n - f) \cdot g\| + \|f \cdot (g_n - g)\| \\ &\leq \|(f_n - f)\| \cdot \|(g_n - g)\| + \|(f_n - f)\| \cdot \|g\| + \|f\| \cdot \|(g_n - g)\| \\ &< 1/(3Mk)^2 + M \cdot 1/(3Mk) + M \cdot 1/(3Mk) < 1/k \end{aligned}$$

Since we can do this for every positive integer k, the conclusion follows.

2. The following questions are about multiplicative inverses of continuous functions on an interval $[a, b]$.

(1 mark) (a) If $f(x) \neq 0$ for all x in $[a, b]$, then show that there is a positive integer k so that $|f(x)| > 1/k$ for *all* x in $[a, b]$.

Solution: As seen in the notes, there is a c in $[a, b]$ so that $|f(c)| \leq |f(x)|$ for all x in $[a, b]$. Since $f(c) \neq 0$, there is a positive integer k so that $|f(c)| > 1/k$. The required result follows.

(1 mark) (b) If $f(x) \neq 0$ for all x in $[a, b]$, then show that $1/f$ is continuous in $[a, b]$ and $\|(1/f)\| < k$ with k as in the previous part.

Solution: Given that $(x_n)_{n\geq 1}$ is a sequence in $[a, b]$ that converges to x in $[a, b]$, we are given that $(f(x_n))_{n\geq 1}$ converges to $y = f(x)$ (since f is continuous). Since $f(x_n) \neq 0$ for all n, the sequence $(1/f(x_n))_{n\geq 1}$ is well-defined. Moreover, since $y = f(x) \neq 0$, the sequence $(1/f(x_n))_{n\geq 1}$ converges to $1/y = 1/f(x)$ (as seen in the arithmetic of sequences). Hence $1/f$ is continuous.

Since $|f(x)| \geq |f(c)|$ for all x in $[a, b]$, we see that $|1/f(x)| \leq |1/f(c)|$. It follows that $\|(1/f)\| = |1/f(c)|$. The required inequality follows by the previous part.

(1 mark) (c) Give an example of a sequence $(f_n)_{n\geq 1}$ converges to f where $f_n(x) \neq 0$ for *all* n and all x, but $f(x) = 0$ for *some* x in $[a, b]$.

Solution: For every positive integer n, we can define $f_n(x) = 1/n$ for all x in $[a, b]$. It is clear that $f(x) = 0$ for all x in $[a, b]$ and the convergence is uniform.

(1 mark) (d) Give an example as above with the additional condition that $\|f_n\| = 1$ for all n.

Solution: For every positive integer n we take the function f_n which is linear, takes the value 1 and a and the value $1/n$ at b. It is given by

$$f_n(x) = \frac{b - x}{b - a} + \frac{x - a}{n(b - a)}$$

It is clear that $0 \leq f_n(x) \leq 1$ for all x in $[a, b]$. So $\|f_n\| = 1$ since $f_n(a) = 1$.

One shows easily that $(f_n)_{n\geq 1}$ converges uniformly to the function $f(x) = (b - x)/(b - a)$ which has the property that $f(b) = 0$.

(1 mark) (e) If $(f_n)_{n\geq 1}$ converges to f and $f(x) \neq 0$ for all x in $[a, b]$ show that there are positive integers p and k so that $|f_n(x)| > 1/k$ for all x in $[a, b]$ and for all $n \geq p$.

Solution: As seen above there is a positive integer m so that $|f(x)| > 1/m$ for all x in $[a, b]$.

Since $(f_n)_{n\geq 1}$ converges to f, there is a positive integer p so that $\|f_n - f\| < 1/(2m)$ for all $n \geq p$. This means that $|f_n(x) - f(x)| < 1/2m$ for all x in $[a, b]$. This gives, for $n \geq p$

$$1/m < |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq 1/(2m) + |f_n(x)|$$

As a result we get, for $n \geq p$

$$|f_n(x)| \geq 1/m - 1/(2m) = 1/(2m)$$

So we get the result required by taking $k = 2m$.

(1 mark) (f) If $(f_n)_{n\geq 1}$ converges to f and $f(x) \neq 0$, then with p as in the previous part, show that $(1/f_n)_{n\geq p}$ converges to $1/f$.

Solution: By the previous part, we have positive integers p and k so that $|f_n(x)| > 1/k$ for all x in $[a,b]$ and for all $n \geq p$. Since $(f_n(x))_{n\geq 1}$ converges to $f(x)$ it follows that $|f(x)| \geq 1/k$ for all x in $[a,b]$. Secondly, we see that $(1/f_n)$ is well defined for $n \geq p$ and as seen above it is continuous as well; similarly $1/f$ is also well-defined and continuous. We have, for all $n \geq p$ and for all x in $[a,b]$

$$\left|\frac{1}{f_n(x)} - \frac{1}{f(x)}\right| = \frac{|f(x) - f_n(x)|}{|f_n(x)||f(x)|} \leq (k^2)\|f_n - f\|$$

It follows that

$$\|(1/f_n) - (1/f)\| \leq k^2\|f_n - f\|$$

Since $(f_n)_{n\geq 1}$ converges to f, it follows easily that the left-hand side can be made as small as one likes by choosing a large enough n. Hence the result follows.