Analysis in One Variable
MTH102 | Second Mid-Sem Exam | 14 March 2019

Solutions

(2 marks) 1. Does the following sequence $(x_n)_{n\geq 1}$ converge and what is the limit in case it converges?

$$x_n = \frac{3n^2 + n + 1}{2^n + 1}$$

Solution: For any positive integer k we have shown that, there is an n_k so that $k(3n^2 + n + 1) < 2^n + 1$ for all $n \geq n_0$. It follows that $x_n < 1/k$ for all $n \geq n_k$. It follows that $(x_n)_{n\geq 1}$ converges to 0.

2. Which of the following functions is continuous in the interval $[0, 1]$?

(2 marks) (a)

$$f(x) = \begin{cases} 0 & \text{for } x < 1/2 \\ 1 & \text{for } x \geq 1/2 \end{cases}$$

Solution: If $x_n = (1/2) + (-1)^n/n$, then $f(x_n) = 0$ for n odd and $f(x_n) = 1$ for n even. Thus, $(f(x_n))_{n\geq 1}$ does not converge. However, $(x_n)_{n\geq 1}$ converges to $(1/2)$. Hence, the function is not continuous.

(2 marks) (b)

$$f(x) = \begin{cases} x & \text{for } x < 1/2 \\ 1 - x & \text{for } x \geq 1/2 \end{cases}$$

Solution: The function $g(x) = x$ is continuous on $[0, 1/2]$ and the function $h(x) = 1 - x$ is continuous on $[1/2, 1]$. Also $g(1/2) = h(1/2)$. The given function f is obtained by "patching" hence it is continuous.

3. Which of the following power series converges uniformly in the region $|x| \leq 1$?

(2 marks) (a) $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

Solution: If $|x| \leq 1$, then $|x^k/2^k| \leq 1/2^k < 1/r^k$ for any $1 < r < 2$. It follows that the power series converges uniformly in this region.
Alternative solution. We have $\limsup|(1/2^k)|^{1/k} = 1/2$. Hence, the series converges uniformly for $|x| \leq r$ for any r such that $(1/r) > (1/2)$; in particular for $r = 1$.

(2 marks) (b) $\sum_{k=1}^{\infty}(-1)^k \frac{x^k}{k}$

> **Solution:** We put $x = -1$ to get the series $\sum_{k=1}^{\infty}(1/k)$ which is divergent. Hence, this series does not converge uniformly in $|x| \leq 1$.

(1 mark) 4. (a) Find a function f on $[0, 1]$ such that $I(f, [0, y])$ is $3y - y^3$ for y such that $0 < y < 1$. (Here $I(f, [0, y])$ denotes the integral of $f(x)$ from 0 to y as defined in the notes.)

> **Solution:** By the result proved in the notes, the integral of $f(x) = 3 - 3x^2$ on the interval $[0, y]$ is $3y - y^3$.

(2 marks) (b) Is $3y - y^3$ increasing or decreasing or neither on the interval $[0, 1]$.

> **Solution:** We note that $3 - 3x^2 \geq 0$ for $0 < x < 1$. It follows that $I(f, [0, y])$ is increasing for y in the interval $[0, 1]$.

(2 marks) (c) Given c such that $0 < c < 2$ is there a unique solution to the equation $y^3 - 3y + c = 0$ with y in the interval $[0, 1]$?

> **Solution:** We have shown that $g(y) = 3y - y^3$ is increasing for y in $[0, 1]$. Since it is continuous it takes every value in $[g(0), g(1)]$ for exactly one value of y; we note that $[g(0), g(1)]$ is the interval $[0, 2]$.