Integration

Following the discussion in earlier sections, we now give a formal definition of the integral of a continuous function. It will exhibit the following properties:

1. Given a continuous function f on $[a, b]$, we will associate a number $I(f, [a, b])$ called the integral of f from a to b.
2. If f is a non-negative function on $[a, b]$, then the integral will be non-negative. If, in addition, f is positive at *some* point of $[a, b]$, then the integral will be positive.
3. If c is a point in the interval $[a, b]$, then we will have the identity $I(f, [a, b]) = I(f, [a, c]) + I(f, [c, b])$.
4. Given a positive number p, if we define $g(x) = f(px)$ as a function on $[a/p, b/p]$, then we have $I(g, [a/p, b/p]) = (1/p)I(f, [a, b])$.
5. Given two continuous functions f and g on $[a, b]$ and a constant c, we have $I(cf + g, [a, b]) = cI(f, [a, b]) + I(g, [a, b])$.
6. Given a linear function $f(x) = c + dx$ in $[a, b]$, the integral is given by the area of the trapezium. In other words $I(f, [a, b]) = (f(a) + f(b))(b - a)/2$.

In fact, the above properties *uniquely* determine the integral!

Piecewise linear functions

Given a continuous function f on $[a, b]$ that piecewise linear with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$. In other words, for t in $[0, 1]$ and $x = (1 - t)x_{i-1} + tx_i$ we have $f(x) = (1 - t)f(x_{i-1}) + tf(x_i)$. Applying Rule 3 (as above) repeatedly, we have:

$$I(f, [a, b]) = I(f, [x_0, x_1]) + I(f, [x_1, x_2]) + \cdots + I(f[x_{n-1}, x_n]) = \sum_{i=1}^{n} I(f, [x_{i-1}, x_i])$$

Moreover, we note that for x in the interval $[x_{i-1}, x_i]$

$$x = (1 - t)x_{i-1} + tx_i \text{ where } t = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

A simple calculation shows that for x in the interval $[x_{i-1}, x_i]$,

$$f(x) = (1 - t)f(x_{i-1}) + tf(x_i) = \frac{f(x_{i-1})x_i - f(x_i)x_{i-1} + x\left(f(x_i) - f(x_{i-1})\right)}{x_i - x_{i-1}}$$

In other words, it is ripe for an application of Rule 6. Hence, we have

$$I(f, [x_{i-1}, x_i]) = \frac{f(x_{i-1} + f(x_i)}{2}\left(x_i - x_{i-1}\right)$$

Combining the two formulas for integrals we get

$$I(f,[a,b]) = \sum_{i=1}^{n} \frac{f(x_{i-1} + f(x_i)}{2}(x_i - x_{i-1}) = T(f,(x_0,x_1,\ldots,x_n))$$

which is the trapezoidal rule for the integral of a piecewise linear continuous function on the interval.

Properties for trapezoidal rule

It is relatively easy to verify all the Rules (1)-(6) for the values obtained by the Trapezoidal rule applied to piecewise linear continuous functions. In fact, we have already done this in earlier sections.

General Continuous function

Given continuous functions f and g on the interval $[a,b]$ so that $f(x) \geq g(x)$ for all x, we can apply Rule 2 to conclude that

$$I(f-g,[a,b]) \geq 0 \text{ since } (f-g)(x) = f(x) - g(x) \geq 0$$

Secondly, by applying Rule 5 to $f - g = (-1)g + f$ to obtain

$$I(f-g,[a,b]) = (-1)I(g,[a,b]) + I(f,[a,b]) = I(f,[a,b]) - I(g,[a,b])$$

It follows that $I(f,[a,b]) \geq I(g,[a,b]$.

In particular, if f is a continuous function, then we have the above inequality for *all* g is a piecewise linear continuous functions which satisfy $f(x) \geq g(x)$ for x in the interval $[a,b]$.

It is thus tempting to expect that $I(f,[a,b])$ is the *supremum* of all the $I(g,[a,b])$; the latter can be calculated using the trapezoidal rule above. Hence, this would give a way to calculate $I(f,[a,b])$ for a *general* continuous function. There are two problems that need to be resolved for this to work:

- We need to prove that the collection of values obtained by the trapezoidal rule for such piecewise linear continuous functions is bounded above.
- We need to extend the notion of supremum to bounded collections of numbers. (We only introduced the notion of supremum for a *sequence* of numbers.)

First of all, we have seen that if f is a continuous function on $[a,b]$, then the values $f(x)$ are bounded for x in this interval. In other words, there is a constant M so that $f(x) \leq M$ for all x in the interval. It follows that, if g is *any* piecewise linear continuous function on $[a,b]$ which satisfies $g(x) \leq f(x)$ for x in the interval $[a,b]$, then $g(x) \leq M$ for all x in the interval. We now see that

$$I(g,[a,b]) = T(g,(x_0,x_1,\ldots,x_n)) = \sum_{i=1}^{n} \frac{g(x_{i-1} + g(x_i)}{2}(x_i - x_{i-1}) \leq M(b-a)$$

for any suitable partition of $[a, b]$ into intervals $[x_{i-1}, x_i]$ on which g is linear. This shows that the collection of values of trapezoidal sums associated with piecewise linear continuous functions dominated by f on $[a, b]$ is bounded above.

Supremum of a bounded set

Given a set A of numbers, which is bounded above, we want to show that there is a supremum. By definition, this is a number b so that $a \leq b$ for every a in A *and*, if c is a number so that $a \leq c$ for every a in A, then $b \leq c$ (so that b is a "least upper bound" of A in some sense).

We proceed by the bisection method. Let x_1 be some element of A and y_1 be some upper bound of A. We define $z_n = (x_n + y_n)/2$ in what follows. In each case: - Either $a \leq z_n$ for all a in A. In this case, we put $y_{n+1} = z_n$ and $x_{n+1} = x_n$. - Or, there is an a in A so that $a > z_n$. In this case, we put $x_{n+1} = a$ and $y_{n+1} = y_n$. We note that $a \leq y_n$ for all n and for all a in A. Similarly, we note that x_n lies in A for all n.

By the usual method of bisection one concludes that $(x_n)_{\neq 1}$ is non-decreasing and $(y_n)_{n\geq 1}$ is non-increasing; moreover, $(y_n - x_n)_{n\neq 1}$ is a sequence of positive numbers decreasing to 0. It follows that $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ converge to the same number c; in fact $(x_n)_{n\geq 1}$ *increases* to c and $(y_n)_{n\geq 1}$ *decreases* to c.

Since $a \leq y_n$ for all n and for all a in A, we see that $a \leq c$ for all a in A. On the other hand, if $a \geq b$ for all a in A, then, in particular $x_n \leq b$ for all n and so $c \leq b$. This shows that c has the desired properties.

Trapezoidal Rule

In the previous subsection we have seen that $I(f, [a, b])$ is the supremum of the integrals $I(g, [a, b])$ where g runs over piecewise linear continuous functions on $[a, b]$ with $f(x) \geq g(x)$ for all x in this interval. While this is useful for *theoretical* questions, from a practical standpoint the following is more useful.

For each positive integer n, let $x_i = a + (b - a)(i/n)$ so that $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a, b]$. We consider the piecewise linear continuous function f_n on $[a, b]$ which is linear on $x_{i-1}, x_i]$ for $i = 1, \ldots, n$ and satisfies $f_n(x_i) = f(x_i)$; in other words, f_n *interpolates* the values of f at x_i. By the formula above, we have

$$I(f_n, [a, b]) == T\left(f_n, (x_0, x_1, \ldots, x_n)\right) = \sum_{i=1}^{n} \frac{f(x_{i-1} + f(x_i)}{2} (x_i - x_{i-1})$$

(Note that it is $f(x_i)$ which appear on the right-hand side.)

By the continuity of f, given a positive integer k, we can choose n so that $|f(x) - f(y)| < 1/k$ whenever $|x - |y < (b - a)/n$ for x, y in $[a, b]$. In earlier sections, we have seen that this means that

$$|f(x) - f_n(x)| < 1/k \text{ for all } x \text{ in } [a, b]$$

In particular, we see that

$$f_n(x) - 1/k < f(x) < f_n(x) + 1/k$$

Thus,

$$I(f_n - 1/k, [a,b]) < I(f, [a,b]) < I(f_n + 1/k, [a,b])$$

We easily calculate

$$I(f_n + 1/k, [a,b]) - I(f_n - 1/k, [a,b]) = 2(b-a)/k$$

Since $I(f, [a,b])$ and $I(f_n, [a,b])$ lie in the interval

$$\left[I\left(f_n - 1/k, [a,b]\right), I\left(f_n + 1/k, [a,b]\right)\right]$$

it easily follows that

$$|I(f, [a,b]) - I(f_n, [a,b])| \leq 2(b-a)/k$$

Thus, by suitably choosing k, and consequently choosing n, we can make this as small as we like.

This gives us the trapezoidal rule for integration

$$I(f, [a,b]) = \lim \left(T\left(f, a, b, n\right)\right)_{n \geq 1}$$

where

$$T(f, a, b, n) = \frac{b-a}{2n} \sum_{i=1}^{n} \left(f\left(a + \frac{(b-a)(i-1)}{n}\right) + f\left(a + \frac{(b-a)i}{n}\right)\right)$$

Convergence

More generally, one can obtain convergence of integrals in terms of the distance between continuous functions (as defined earlier),

$$\|f - g\|_{[a,b]} = \sup\left\{|f(x) - g(x)| : x \text{ in } [a,b]\right\}$$

We have the inequality

$$f - |f - g|_{[a,b]} \leq g \leq f + |f - g|_{[a,b]}$$

Applying the Rules given above we easily prove

$$|I(f, [a,b]) - I(g, [a,b])| \leq \|f - g\|_{[a,b]}(b-a)$$

As a consequence, one sees that if $(f_n)_{n \geq 1}$ is a sequence of continuous functions converging uniformly on the interval $[a,b]$ to a continuous function f, then the sequence $(I(f_n, [a,b]))_{n \geq 1}$ of integrals converges to $I(f, [a,b])$.

Integrals of polynomials and power series

It has been seen by direct computation that if $f(x) = a_0 + a_1 x + \cdots + a_p x^p$ is a polynomial function, then

$$I(f, [0, y]) = I(a_0 + a_1 x + \cdots + a_p x^p, [0, y]) = a_0 y + a_1 \frac{y^2}{2} + \cdots + a_p \frac{y^{p+1}}{p+1}$$

Moreover, by the above result on uniform convergence, we can extend this to power series as follows. Assume that the power series $\sum_{k=0}^{\infty} a_n x^n$ has radius of convergence R. This means that the sequence of partial sums (which are polynomial functions) converge uniformly for x such that $|x| \leq r < R$, for all $0 < r < R$. Thus, given y such that $0 < y < R$, it follows that

$$I(\sum_{k=0}^{\infty} a_n x^n, [0, y]) = \sum_{k=0}^{\infty} a_n \frac{y^{n+1}}{n+1}$$

and the power series on the right hand side *also* converges uniformly for y such that $|y| \leq r < R$ for all $0 < r < R$.

Standard Notation

Given the above results, which are consistent with the rules of integration found in calculus, it is more natural to use the notation

$$\int_a^b f(x)dx = I(f, [a, b])$$

This notation is also a convenient way to directly express Rule 4 since we get, (a special case of) the change of variable formula for $p > 0$

$$\int_{a/p}^{b/p} f(px)dx = (1/p) \int_a^b f(t)dt$$

Secondly, we can also make sense of $\int_a^b f(x)dx$ for $a > b$ by defining

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

This is easily seen to be consistent with Rules 3 and 6. Thus, we get a change of variable formula for *all* non-zero constants p

$$\int_a^b f(t)dt = p \int_{a/p}^{b/p} f(px)dx$$

This notation is much more convenient than the notation $I(f, [a, b])$ and thus we will use it in future.