Uniform Convergence

1. The following questions are about continuous functions on an interval $[a, b]$ and convergence is with respect to $\|\cdot\|_{[a,b]}$ (in other words, uniform convergence).

(1 mark) (a) Show that
$$\|f + g\| \leq \|f\| + \|g\|$$

(1 mark) (b) Show that
$$\|f \cdot g\| \leq \|f\| \cdot \|g\|$$

(1 (bonus)) (c) Find an example of non-zero functions f and g on $[0, 1]$ such that $f \cdot g = 0$.

(1 mark) (d) If $(f_n)_{n\geq 1}$ converges to f and $(g_n)_{n\geq 1}$ converges to g, then show that $(f_n + g_n)_{n\geq 1}$ converges to $f + g$.

(1 mark) (e) If $(f_n)_{n\geq 1}$ converges to f and $(g_n)_{n\geq 1}$ converges to g, then show that $(f_n \cdot g_n)_{n\geq 1}$ converges to $f \cdot g$.

2. The following questions are about multiplicative inverses of continuous functions on an interval $[a, b]$.

(1 mark) (a) If $f(x) \neq 0$ for all x in $[a, b]$, then show that there is a positive integer k so that $|f(x)| > 1/k$ for *all* x in $[a, b]$.

(1 mark) (b) If $f(x) \neq 0$ for all x in $[a, b]$, then show that $1/f$ is continuous in $[a, b]$ and $\|(1/f)\| < k$ with k as in the previous part.

(1 mark) (c) Give an example of a sequence $(f_n)_{n\geq 1}$ converges to f where $f_n(x) \neq 0$ for *all* n and all x, but $f(x) = 0$ for *some* x in $[a, b]$.

(1 mark) (d) Give an example as above with the additional condition that $\|f_n\| = 1$ for all n.

(1 mark) (e) If $(f_n)_{n\geq 1}$ converges to f and $f(x) \neq 0$ for all x in $[a, b]$ show that there are positive integers p and k so that $|f_n(x)| > 1/k$ for all x in $[a, b]$ and for all $n \geq p$.

(1 mark) (f) If $(f_n)_{n\geq 1}$ converges to f and $f(x) \neq 0$, then with p as in the previous part, show that $(1/f_n)_{n\geq p}$ converges to $1/f$.