Computing integrals of polynomials

The trapezoidal rule for integration can be applied to the function x^p on the interval $[0, 1]$ with the partition $0 < (1/n) < (2/n) < \cdots < (n/n) = 1$ to give the approximation

$$I_n = \sum_{k=1}^{n} \frac{(k-1)^p + k^p}{2n^p} \cdot \frac{1}{n}$$

of the integral $I(x^p, [0, 1])$. In this section we will calculate the limit of I_n and use it to calculate the integral $I(f(x), [a, b])$ where f is *any* polynomial function and $[a, b]$ is *any* interval.

A summation identity for binomial functions

We have seen the following fundamental identity between binomial coefficients

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$$

Next, we note the elementary identity

$$\sum_{k=0}^{n} \binom{k}{0} = \binom{n+1}{1}$$

More generally, for a fixed positive integer r, we now claim a proof by induction on n that

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

When $n = 0$, this becomes $\binom{0}{r} = \binom{1}{r+1}$ in which both sides are 0 since $r \geq 1$. Suppose that we are given this identity for $n-1$ for some positive integer n. Then we have

$$\sum_{k=0}^{n} \binom{k}{r} = \left(\sum_{k=0}^{n-1} \binom{k}{r}\right) + \binom{n}{r} = \binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}$$

where we used the induction hypothesis for the second equality.

Sums of powers as a polynomial function

We will now use this to prove, by induction on p, that

$$\sum_{k=0}^{n} k^p = \frac{n^{p+1}}{p+1} + f_p(n)$$

where $f_p(n)$ is a polynomial of degree at most p in the variable n. First of all, let us note that when $p = 0$, we have the identity

$$\sum_{k=0}^{n} k^0 = n + 1 = \frac{n^{0+1}}{0+1} + 1$$

and 1 is a polynomial of degree 0. So we have the result for $p = 0$. Now let us assume the result for all $q < p$ for some positive integer p. We note that

$$\binom{k}{p} = \frac{k(k-1)\cdots(k-p+1)}{p!} = \frac{k^p}{p!} + g_{p-1}(k)$$

where $g_{p-1}(k)$ is a polynomial function of k of degree *less than* p. If

$$g_{p-1}(k) = a_0 k^0 + a_1 k + \cdots + a_{p-1} k^{p-1}$$

then, we apply the induction hypothesis to

$$h_p(n) = \sum_{k=0}^{n} g_{p-1}(k) = a_0 \sum_{k=0}^{n} k^0 + a_1 \sum_{k=0}^{n} k + \cdots + a_{p-1} \sum_{k=0}^{n} k^{p-1}$$

to realise that $h_p(n)$ is a polynomial of degree *at most* p in the variable n. It follows that

$$\sum_{k=0}^{n} \binom{k}{p} = \frac{1}{p!}\left(\sum_{k=0}^{n} k^p\right) + h_p(n)$$

Using the above summation identity for binomial coefficients, we get

$$\binom{n+1}{p+1} = \frac{1}{p!}\left(\sum_{k=0}^{n} k^p\right) + h_p(n)$$

Now, as above

$$\binom{n+1}{p+1} = \frac{(n+1)n(n-1)\cdots(n-p+1)}{(p+1)!} = \frac{n^{p+1}}{(p+1)!} + g_p(n)$$

where $g_p(n)$ is a polynomial of degree at most p in the variable n. Combining these identities, we get

$$\frac{1}{p!}\left(\sum_{k=0}^{n} k^p\right) + h_p(n) = \frac{n^{p+1}}{(p+1)!} + g_p(n)$$

where both h_p and g_p are polynomials of degree at most p in the variable n. Multiplying by $p!$ and moving $h_p(n)$ to the other side, we obtain

$$\sum_{k=0}^{n} k^p = \frac{n^{p+1}}{p+1} + p!g_p(n) - h_p(n)$$

Hence, we have the identity of the required type with $f_p(n) = p!g_p(n) - h_p(n)$.

Estimation of I_n

We use the identity proved above to write

$$\sum_{k=1}^{n} \frac{(k-1)^p + k^p}{2n^p} \cdot \frac{1}{n} = \sum_{k=0}^{n-1} \frac{k^p}{2n^{p+1}} + \sum_{k=0}^{n} \frac{k^p}{2n^{p+1}} = \frac{(n-1)^{p+1} + n^{p+1}}{2(p+1)n^{p+1}} + \frac{a_p(n)}{2n^{p+1}}$$

where $a_p(n)$ is a polynomial of degree at most p in the variable n.

By applying results proved in the section on polynomial growth, we see that

$$\lim \left(\frac{a_p(n)}{2n^{p+1}} \right)_{n \geq 1} = 0$$

It follows that

$$\lim(I_n)_{n \geq 1} = \lim \left(\frac{(n-1)^{p+1} + n^{p+1}}{2(p+1)n^{p+1}} \right)_{n \geq 1} = \frac{1}{p+1}$$

Thus, we obtain $I(x^p, [0,1]) = 1/(p+1)$ as required.

Scaling in axial directions

Suppose f is a piecewise linear continuous function on $[a,b]$ which is linear with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$. Given positive constants p and q and consider the function g given by $g(x) = qf(px)$. Then g is a piecewise linear continuous function on $[a/p, b/p]$ given with respect to the partition $a/p = x_0/p < x_1/p < \cdots < x_n/p = b/p$.

The trapezoidal rule applied to g gives

$$\begin{aligned} I(g, [a/p, b/p]) &= \sum_{k=1}^{n} \frac{g(x_{i-1}/p) + g(x_i/p)}{2} \cdot \frac{x_i - x_{i-1}}{p} \\ &= \sum_{k=1}^{n} \frac{qf(x_{i-1}) + qf(x_i)}{2} \cdot \frac{x_i - x_{i-1}}{p} = (q/p) I(f, [a,b]) \end{aligned}$$

For a general continuous function h, the integral $I(h, [a,b])$ is defined as a limit of integrals of piecewise linear continuous approximations of h. Hence we deduce > Given a continuous function f on $[a,b]$ and positive constants p and q, let g be the continuous function on $[a/p, b/p]$ defined by $g(x) = qf(px)$, then we have the identity $I(g, [a/p, b/p]) = (q/p)I(f, [a,b])$

Integrals of sums of functions

If f and g are piecewise linear continuous functions on $[a,b]$ which are linear with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$, then it is clear that

$f + g$ is also linear with respect to the same partition. Moreover, we calculate

$$I(f+g,[a,b]) = \sum_{k=1}^{n} \frac{f(x_{i-1}) + g(x_{i-1}) + (f(x_i) + g(x_i)}{2}(x_i - x_{i-1})$$
$$= I(f,[a,b]) + I(g,[a,b])$$

The integrals of general continuous functions are defined as limits of piecewise linear continuous approximations of the functions. Moreover, suppose f_1 is an approximation of f with error at most $1/p$ and g_1 is an approximation of g with error at most $1/q$. Then $f_1 + g_1$ is an approximation of $f + g$ with error at most $(1/p) + (1/q)$. Hence, we deduce > Given continuous functions f and g on $[a, b]$ we have the identity $I(f + g, [a, b]) = I(f, [a, b]) + I(g, [a, b])$.

Integrals of polynomials

We now combine the results of the previous three sections. First of all we get

$$I(a_0x^0 + a_1x^1 + \cdots + a_nx^n, [0,y]) = \sum_{k=0}^{n} I(a_kx^k, [0,y])$$

Next, we note that if $q = a_ky^k$ and $p = 1/y$, then the function $g(x) = a_kx^k$ on $[0, y]$ is obtained from the function $f(x) = x^k$ on $[0, 1]$ by the identity $g(x) = qf(px)$. It follows that

$$I(a_kx^k, [0,y]) = \frac{a_ky^k}{(1/y)} I(x^k, [0,1]) = a_k\frac{y^{k+1}}{k+1}$$

where we have also used the computation of $I(x^k, [0, 1])$.

Combining these identities we see that

$$I(a_0x^0 + a_1x^1 + \cdots + a_nx^n, [0,y]) = a_0\frac{y^1}{1} + a_1\frac{y^2}{2} + \cdots + a_n\frac{y^{n+1}}{n+1}$$

This completes the proof of a well-known formula in high-school calculus!