Solutions to Assignment 8

(2 marks) 1. For each $a > 0$ show that there is a unique positive solution $f(a)$ of the equation $x^3 + x = a$. Moreover, show that the function f defined as $a \mapsto f(a)$ is a continuous function.

> **Solution:** For any positive integer n, the function g on the interval $[0, n]$ defined by $g(x) = x^3 + x$ is continuous and increasing (it is the sum of two increasing functions). Hence, as shown in the notes it has a continuous inverse on the interval $[g(0), g(n)]$. (1 Mark for this part).
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> Since $g(0) = 0$ and $g(n) > n$, given any positive number a it lies in a interval of the form $[g(0), g(n)]$ for some a (Archimedean principle). (1 Mark for this part)

(3 marks) 2. For a positive continuous function f on the interval $[a, b]$, let $I(f, [a, b])$ denote the area of the region $A(f)$ as described in the notes. For any positive numbers p and q, let us define the function g on $[a/p, b/p]$ by $g(x) = qf(px)$. What is $I(g, [a/p, b/p])$ in terms of $I(f, [a, b])$? Justify your answer. (Hint: First examine the case when the region is a trapezium.)

> **Solution:** Compare the trapezium with corners $(a, 0)$, (a, c), (b, d) and $(b, 0)$ with the trapezium with corners $(a/p, 0)$, $(a/p, qc)$, $(b/p, qd)$ and $(b/p, 0)$. The areas are
>
> $$\frac{(c+d)(b-a)}{2} \text{ and } \frac{q(c+d)(b-a)}{2p}$$
>
> (1 Mark for this calculation.)
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> In other words, scaling by $1/p$ in the x-direction and by q in the y-direction in the plane leads to scaling of the area of a trapezium by q/p. Since the areas we are calculating are limits of sums of areas of such trapeziums, the same applies to all such areas. (1 Mark for this observation.)
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> Since the region $A(g)$ is obtained from the region $A(f)$ by such a scaling we see that $I(g, [a/p, b/p]) = (q/p)I(f, [a, b])$. (1 Mark for this calculation).

3. By the trapezoidal rule we get an approximation of $I(x^2, [0, 1])$ as

$$I_n = \sum_{k=0}^{n-1} \frac{k^2 + (k+1)^2}{2n^2} \frac{1}{n}$$

by using the partition $0 < 1/n < 2/n < \cdots < 1$. In the following sequence of exercises we show that the sequence $(I_n)_{n \geq 1}$ converges to $1/3$.

(1 mark) (a) Show the identity

$$\binom{n+1}{r+1} - \binom{n}{r+1} = \binom{n}{r}$$

Solution: We have

$$\begin{aligned}\binom{n+1}{r+1} - \binom{n}{r+1} &= \frac{(n+1)!}{(r+1)!(n-r)!} - \frac{n!}{(r+1)!(n-r-1)!} \\ &= \frac{n!}{(r+1)!(n-r)!}\left((n+1)-(n-r)\right) \\ &= \frac{n!}{(r+1)!(n-r)!}(r+1) = \binom{n}{r}\end{aligned}$$

(1 mark) (b) Show the identity (use induction on n)

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

for all $r \geq 0$.

Solution: For a fixed r, let $a(n)$ denote the left-hand side and $b(n)$ denote the right-hand side. For $r = 0$, $a(0) = 1$ and $b(0) = 1$ and for $r \geq 1$, $a(0) = 0$ and $b(0) = 0$. So, for $n = 0$ we get the equality of both sides.
The difference $a(n+1) - a(n) = \binom{n+1}{r}$ and the difference

$$b(n+1) - b(n) = \binom{n+2}{r+1} - \binom{n+1}{r+1} = \binom{n+1}{r}$$

by the previous exercise. Thus, given the identity $a(n) = b(n)$ by induction, we obtain $a(n+1) = b(n+1)$.
Hence, the identity $a(n) = b(n)$ is proved for all n by induction.

(1 mark) (c) Show the identities

$$\sum_{k=0}^{n} k = \binom{n+1}{2}$$

$$\sum_{n=0}^{n} (k^2 - k) = 2\binom{n+1}{3}$$

Solution: The previous exercise, for the case $r = 1$ gives

$$\sum_{k=0}^{n} k = \sum_{k=0}^{n} \binom{k}{1} = \binom{n+1}{2}$$

Similarly, for the case $r = 2$ it gives

$$\sum_{k=0}^{n} (k^2 - k) = \sum_{k=0}^{n} 2\binom{k}{2} = 2\binom{n+1}{3}$$

(2 marks) (d) Use the identities above to calculate the limit of I_n.

Solution: We have

$$\sum_{k=0}^{n-1} k^2 = \sum_{k=0}^{n-1} (k^2 - k) + \sum_{k=0}^{n-1} k = 2\binom{n}{3} + \binom{n}{2}$$

Hence, we have

$$I_n = \frac{1}{2n^3}\left(2\binom{n}{3} + \binom{n}{2} + 2\binom{n+1}{3} + \binom{n+1}{2}\right)$$

(1 Mark for this observation.)
We now calculate

$$I_n = \frac{2n(n-1)(n-2)}{2n^3(3!)} + \frac{n(n-1)}{2n^3(2!)} + \frac{2(n+1)n(n-1)}{2n^3(3!)} + \frac{(n+1)n}{2n^3(2!)}$$
$$= \frac{(1-(1/n))(1-(2/n))}{6} + \frac{(1-(1/n))}{4n} + \frac{(1+(1/n))(1-(1/n))}{6} + \frac{(1+(1/n))}{4n}$$

It is now clear the the limit of $(I_n)_{n\geq 1}$ is $1/6 + 1/6 = 1/3$. (1 Mark for this calculation.)