Monotone Functions and Areas

(2 marks) 1. For each $a > 0$ show that there is a unique positive solution $f(a)$ of the equation $x^3 + x = a$. Moreover, show that the function f defined as $a \mapsto f(a)$ is a continuous function.

(3 marks) 2. For a positive continuous function f on the interval $[a, b]$, let $I(f, [a, b])$ denote the area of the region $A(f)$ as described in the notes. For any positive numbers p and q, let us define the function g on $[a/p, b/p]$ by $g(x) = qf(px)$. What is $I(g, [a/p, b/p])$ in terms of $I(f, [a, b])$? Justify your answer. (Hint: First examine the case when the region is a trapezium.)

3. By the trapezoidal rule we get an approximation of $I(x^2, [0, 1])$ as

$$I_n = \sum_{k=0}^{n-1} \frac{k^2 + (k+1)^2}{2n^2} \frac{1}{n}$$

by using the partition $0 < 1/n < 2/n < \cdots < 1$. In the following sequence of exercises we show that the sequence $(I_n)_{n\geq 1}$ converges to $1/3$.

(1 mark) (a) Show the identity

$$\binom{n+1}{r+1} - \binom{n}{r+1} = \binom{n}{r}$$

(1 mark) (b) Show the identity (use induction on n)

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

(1 mark) (c) Show the identities

$$\sum_{k=0}^{n} k = \binom{n+1}{2}$$

$$\sum_{n=0}^{n} (k^2 - k) = 2\binom{n+1}{3}$$

(2 marks) (d) Use the identities above to calculate the limit of I_n.