Areas

Using the Archimedean model of the number system allows us to assign numbers to lengths. What about areas, volumes, angles and so on? How do we use the least upper bound principle to assign numbers to such measurements?

Some high-school Geometry

We will use some basic ideas about areas as learnt in high-school geometry.

- The area $|T|$ of a trapezium T with corners $(a, 0)$, (a, p), (b, q), $(b, 0)$ is given by the formula $|T| = (b-a)(p+q)/2$. We note that this formula is correct for all positive numbers p and q and for all numbers a and b so that $b > a$.
- The area of the union of two regions which have *at most* a line segment in common is the sum of the area of the two regions.
- The area of a region is the same as the area of any translate of the region.

We will use three three ingredients in our calculations below. The fact that there is a notion of area for (some) planar regions that can consistently be made to satisfy these "axioms" is something that needs to be separately justified; we will not do so here!

One particular aspect of justification is as follows. Given a trapezium T with corners $(a, 0)$, (a, p), (b, q), $(b, 0)$. Suppose we pick a number c between a and b. The point vertically above it on the opposite edge in T is (c, r) where $r = (p(b-c) + q(c-a))/(b-a)$. We "break" T along this vertical segment into T_1 and T_2, the left and right piece respectively. We calculate that the sum of the areas of T_1 and T_2 is

$$
\begin{aligned}
&\frac{(c-a)(p+r)}{2} + \frac{(b-c)(r+q)}{2} \\
&\quad = \frac{(c-a)(p+q)}{2} - \frac{(c-a)(q-r)}{2} + \frac{(b-c)(p+q)}{2} - \frac{(b-c)(r-p)}{2} \\
&\quad = \frac{(b-a)(p+q)}{2} - (c-a)\frac{(q-p)(b-c)}{2(b-a)} - (b-c)\frac{(q-p)(c-a)}{2(b-a)} \\
&\quad = \frac{(b-a)(p+q)}{2}
\end{aligned}
$$

In other words, the second condition above is consistent with the first condition when we break a trapezium into two trapeziums: $|T| = |T_1| + |T_2|$.

Area under a graph

Consider the region $A(f)$ described as follows. $A(f)$ is bounded by on three sides by the line segment joining $(a, 0)$, $(b, 0)$, the line segment joining $(a, 0)$, (a, p) and the line segment joining $(b, 0)$, (b, q). On the fourth side that we have the graph of a continuous non-negative function f on $[a, b]$ such that $f(a) = p$ and $f(b) = q$. (Recall that the graph is the locus of points of the form $(x, f(x))$ for x in the interval $[a, b]$.) Can we give a reasonable notion of area $|A(f)|$ for this region?

Area under a broken line

Recall that we say a continuous function f is piecewise linear on $[a, b]$ if there is a finite set of points $a = x_0 < x_1 < \cdots < x_n = b$ so that $f(x) = (1-t)f(x_{i-1}) + tf(x_i)$ when $x = (1-t)x_{i-1} + tx_i$ is a point in the interval $[x_{i-1}, x_i]$ (for $0 \leq t \leq 1$ and i some integer between 1 and n). We first want to compute the area under the graph of such a piecewise linear function, assuming in addition that f is non-negative. This graph can be seen as a "broken line".

In this case, we see that the region $A(f)$ for such a function f is the union of the trapeziums T_i, for $i = 1, \ldots, n$, where T_i is the trapezium described by $(x_{i-1}, 0)$, $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_i, 0)$. The trapeziums T_i and T_{i+1} meet along a common edge and there are no other overlaps. By applying the above axioms for area, we see that the area $|A(f)|$ is the sum of the areas $|T_i|$. In other words, it is

$$(x_1 - x_0) \cdot \frac{f(x_0) + f(x_1)}{2} + (x_2 - x_1) \cdot \frac{f(x_1) + f(x_2)}{2} + \cdots + (x_n - x_{n-1}) \cdot \frac{f(x_n) + f(x_{n-1})}{2}$$

In particular, the area $|A(f)|$ can be easily calculated in terms of the values of the function at the points x_i for $i = 0, \ldots, n$.

Note that if someone *adds* some (a finite number of) points to the list ($a = x_0 < x_1 < \cdots < x_n = b$), then the area computed above does not change! The reason is that adding a point x' between x_{i-1} and x_i is the same as breaking the trapezium T_i along the vertical line through x'.

Comparison of two areas

Suppose that f and g are two piecewise linear functions on $[a, b]$. Suppose that $a = x_0 < x_1 < \cdots < x_n = b$) is a sequence of points chosen so that *both* the functions are *linear* on each interval $[x_{i-1}, x_i]$. In other words, for every t lying between 0 and 1 we have

$$\begin{aligned} f((1-t)x_{i-1} + tx_i) &= (1-t)f(x_{i-1}) + tf(x_i) \\ g((1-t)x_{i-1} + tx_i) &= (1-t)g(x_{i-1}) + tg(x_i) \end{aligned}$$

As above the areas of the regions $A(f)$ and $A(g)$ can be calculated in terms of the areas of trapeziums.

Suppose that for some constant c we have the inequality $|f(x) - g(x)| < c$ for *all* x in the interval $[a, b]$. We calculate,

$$
\begin{aligned}
&|A(f)| - |A(g)| = \\
&\qquad \frac{x_1 - x_0}{2} \cdot ((f(x_0) + f(x_1)) - (g(x_0) + g(x_1))) + \\
&\qquad \frac{x_2 - x_1}{2} \cdot ((f(x_1) + f(x_2)) - (g(x_1) + g(x_2))) + \\
&\qquad \cdots + \\
&\qquad \frac{x_n - x_{n-1}}{2} \cdot ((f(x_{n-1}) + f(x_n)) - (g(x_{n-1}) + g(x_n)))
\end{aligned}
$$

Since *each* of the differences $|f(x_i) - g(x_i)|$ is at most c, we obtain

$$
\begin{aligned}
&||A(f)| - |A(g)|| \leq \\
&\qquad \frac{x_1 - x_0}{2} \cdot (|f(x_0) - g(x_0)| + |f(x_1) - g(x_1)|) + \\
&\qquad \cdots + \\
&\qquad \frac{x_n - x_{n-1}}{2} \cdot (|f(x_{n-1}) - g(x_{n-1})| + |f(x_n) - g(x_n)|) \\
&\qquad \leq (b - a)c
\end{aligned}
$$

Thus, a uniform bound on the difference between the functions gives a uniform bound on the difference between the areas under them.

Area under a general function

Since f is (uniformly) continuous on $[a, b]$ we know that given a positive integer k, there is a positive integer n_k so that $|f(x) - f(y)| \leq 1/k$ if $|x - y| \leq (b - a)/n_k$. So, we can define a function piecewise linear function f_k (as done in an earlier section) as follows. Let r be the integer part of $(b - a)n_k$ and use the notation $a_p = a + p/n_k$ for $p = 0, \ldots, r$ and $a_{r+1} = b$. We define

$$
f_k((1 - t)a_p + ta_{p+1}) = (1 - t)f(a_p) + tf(a_{p+1}) \text{ for } 0 \leq t \leq 1
$$

Given any x lying in the interval $[a_p, a_{p+1}]$ we put $t = (x - a_p)/(a_{p+1} - a_p)$ and note that $x = (1 - t) \cdot a_p + t \cdot a_{p+1}$. So we have defined f_k for all values of x in the interval $[a, b]$. Moreover, we have (for this x and t)

$$
\begin{aligned}
|f(x) - f_k(x)| &= |f(x) - (1 - t)f(a_p) + tf(a_{p+1})| \\
&\leq (1 - t)|f(x) - f(a_p)| + t|f(x) - f(a_{p+1})| < 1/k
\end{aligned}
$$

In other words, as seen earlier, the function f_k *uniformly* approximates f in the interval $[a, b]$ with an error of at most $1/k$.

Suppose that g is another piecewise linear continuous function such that $|g(x) - f(x)| < c$ is a uniform bound on the difference for x in the interval $[a, b]$.

As we have seen the areas $|A(f_k)|$ and $|A(g)|$ are well-defined since these regions are unions of Trapeziums. Moreover, we have $|g(x) - f_k(x)| < c + 1/k$. As seen above, this means that the difference between these areas is

$$||A(f_k)| - |A(g)|| \leq (b-a)(c + 1/k)$$

In particular, if we consider the sequence of functions $(f_k)_{k\geq 1}$, then we have

$$||A(f_k)| - |A(f_l)|| \leq (b-a)(1/l + 1/k)$$

It follows easily, that $(|A(f_k)|)_{k\geq 1}$ is a sequence of numbers satisfying Cauchy's criterion for convergence. Hence, it converges to a number which want to call the area $|A(f)|$ of the region $A(f)$.

To get some more intuition, we can estimate the error in using the area of the region $A(f_k)$ as an approximation to the area $A(f)$. The graph of f over $[a_p, a_{p+1}]$ lies inside the parallelogram with corners $(a_p, f(a_p) - 1/k)$, $(a_p, f(a_p) + 1/k)$, $(a_{p+1}, f(a_{p+1}) + 1/k)$, $(a_{p+1}, f(a_{p+1}) - 1/k)$. The graph of f_k over $[a_p, a_{p+1}]$ also lies in this parallelogram. Hence, the error is *at most* the area of this parallelogram! By using the calculations above, we see that this is $(2/k)(a_{p+1} - a_p)$. Adding up all the errors, we see that the error is at most $(2/k)(b-a)$.

Trapezoidal rule

As a result of the above discussion, we see that the area of the region $A(f)$ is well-defined.

Moreover, to approximate the area of the region $A(f)$ with an error of at most $1/p$, we choose a k so that $(2/k)(b-a) < 1/p$. We then find a "partition" $(a = x_0 < x_1 < \cdots < x_n = b)$ so that for any x, y in the interval $[x_{i-1}, x_i]$ we have $|f(x) - f(y)| < 1/k$. The approximate value of $A(f)$ up to an error of at most $1/p$ is given by

$$(x_1 - x_0) \cdot \frac{f(x_0) + f(x_1)}{2} + (x_2 - x_1) \cdot \frac{f(x_1) + f(x_2)}{2} + \cdots + (x_n - x_{n-1}) \cdot \frac{f(x_n) + f(x_{n-1})}{2}$$

This method of calculating the approximate area of the region $A(f)$ known as the Trapezoidal rule.