Solutions to Assignment 7

(2 marks) 1. Consider the functions $f_n(x) = 2^{2n}x^n(1-x)^n$. Is the sequence (f_n) uniformly convergent in $[0,1]$?

> **Solution:** We note that
>
> $$f_n(x) = 2^{2n}x^n(1-x)^n = y^n$$
>
> where $y = 4x(1-x)$.
>
> If $x = (1/2) - (1/k)$ for some positive integer $k \geq 2$, then
>
> $$x(1-x) = \left(\frac{1}{2} - \frac{1}{k}\right) \cdot \left(\frac{1}{2} + \frac{1}{k}\right) = \frac{1}{4} - \frac{1}{k^2}$$
>
> Hence, $y = 4x(1-x) < 1$. Thus, the sequence $(y^n)_{n\geq 1}$ converges to 0.
>
> It follows that $(f_n(x))_{n\geq 1}$ converges to 0 for $x = (1/2) - (1/k)$ for $k \geq 2$. Note also that $((1/2) - (1/k))_{k\geq 1}$ converges to $(1/2)$.
>
> On the other hand, it is clear than $f_n(1/2) = 1$ for all n.
>
> Thus, the limit function is *not* continuous at $x = (1/2)$. However, the functions $f_n(x)$ *are* continuous functions in $[0,1]$. It follows that $(f_n)_{n\geq 1}$ is not uniformly convergent in any interval containing $(1/2)$.

2. Show that each of the following the series gives a continuous function in $|x| < 1$.

(1 mark) (a) $\sum_{n=0}^{\infty} \frac{1}{n+1}x^n$.

> **Solution:** The $(1/(n+1))_{n\geq 0}$ sequence is bounded by 1. Hence as seen in the notes the power series converges uniformly for $|x| < r$ for any $r < 1$. Thus it gives a continuous function in $|x| < 1$.

(1 mark) (b) $\sum_{n=1}^{\infty} \frac{n}{1+n}x^n$.

> **Solution:** The $(n/(n+1))_{n\geq 1}$ sequence is bounded by 1. Hence as seen in the notes the power series converges uniformly for $|x| < r$ for any $r < 1$. Thus it gives a continuous function in $|x| < 1$.

(1 mark) 3. Show that the series $\sum_{n=1}^{\infty} nx^n$ gives a continuous function in $|x| < 1$.

Solution: If s is a number such that $0 < s < 1$, then we have seen that $n < (1/s)^n$ for sufficently large n.

Hence, as seen in the notes the power series converges uniformly for $|x| \leq r$ for any $r < s$. Thus it gives a continuous function in $|x| < s$.

Since $0 < s < 1$ is arbitrary, it follows that we get a continuous function in $|x| < 1$.

4. For any number α and a positive integer k, we define the function

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha - 1) \cdots (\alpha - k + 1)}{k!}$$

In the following sequence of exercises, we will show that the series

$$f_\alpha(x) = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

converges to a continuous function for $|x| < 1$. This power series is called the (generalised) "binomial series" for α-th power of $(1 + x)$.

(1 mark) (a) Show that if $-1 \leq \alpha < 0$, then for all positive integers k we have

$$\left| \binom{\alpha}{k} \right| \leq 1$$

Solution: We note that $|\alpha| = -\alpha$. Hence

$$|\alpha \cdot (\alpha - 1) \cdots (\alpha - k + 1)| = (-\alpha) \cdot (1 - \alpha) \cdots (k - \alpha - 1) \leq 1 \cdot 2 \cdots k = k!$$

It follows that

$$\left| \binom{\alpha}{k} \right| \leq \frac{k!}{k!} = 1$$

(1 mark) (b) Assuming $\alpha \geq 0$ choose a positive integer r so that $-1 \leq \alpha - r < 0$ (by Archimedean principle!). Then for any positive integer k show that

$$\left| \binom{\alpha}{k} \right| \leq C_r$$

for some constant C_r depending only on r (and not on k).

Solution: We note that $|\alpha - r| = r - \alpha$. Hence

$$\begin{aligned}|(\alpha - r)\cdot(\alpha - r - 1)\cdots(\alpha - k + 1)| &= (r - \alpha)\cdot(r + 1 - \alpha)\cdots(k - \alpha - 1) \\ &\leq r\cdot(r+1)\cdots k = \frac{k!}{(r-1)!}\end{aligned}$$

It follows that for $k > r$, we have

$$\begin{aligned}\left|\binom{\alpha}{k}\right| &= |\alpha\cdot(\alpha-1)\cdots(\alpha - r + 1)|\cdot\frac{|(\alpha - r)\cdot(\alpha - r - 1)\cdots(\alpha - k + 1)|}{k!} \\ &\leq |\alpha\cdot(\alpha-1)\cdots(\alpha - r + 1)|\cdot\frac{k!}{(r-1)!k!} \\ &= \frac{|\alpha\cdot(\alpha-1)\cdots(\alpha - r + 1)|}{(r-1)!}\end{aligned}$$

We take C_r to be the maximum of the above number and $|\binom{\alpha}{k}|$ for $k \leq r$.

(1 mark) (c) Assume that there is a positive integer r so that $-r \leq \alpha < -r + 1$ and show that

$$\left|\binom{\alpha}{k}\right| \leq \binom{r+k-1}{r-1}$$

Solution: We note that $|\alpha| = -\alpha$. Hence

$$\begin{aligned}|\alpha\cdot(\alpha-1)\cdots(\alpha - k + 1)| &= (-\alpha)\cdot(1-\alpha)\cdots(k - \alpha - 1) \\ &\leq r\cdot(r+1)\cdots(r+k-1) = \frac{(r+k-1)!}{(r-1)!}\end{aligned}$$

It follows that

$$\left|\binom{\alpha}{k}\right| \leq \frac{(r+k-1)!}{(r-1)!k!} = \binom{r+k-1}{r-1}$$

(1 mark) (d) For a fixed r show that $\binom{r+k-1}{r-1}$ is a polynomial function of k of degree $r - 1$.

Solution: Note that

$$\binom{r+k-1}{r-1} = \frac{(r+k-1)\cdot(r+k-2)\cdots(k+1)}{(r-1)!}$$

From this expression it is clear that it is a polynomial of degree $(r-1)$ in the variable k.

(2 marks) (e) In each case above, use the results already proved in the notes to conclude that the power series $f_\alpha(x)$ converges to define a continuous function in $|x| < 1$.

Solution: In the first and second case ($\alpha \geq -1$) the parts (1) and (2) above show that the coefficients are bounded independent of k.

In case (3), the coefficents are bounded by a polynomial function of k. Thus, for any s such that $0 < s < 1$, the coefficients are dominated by $((1/s)^k)_{k\geq 1}$. By the results proved in the notes, the series is uniformly convergent in $|x| \leq r$ for every $r < s$. Since s is also arbitrary, we get this for all r with $0 < r < 1$.

It follows that for every α, the series defining $f_\alpha(x)$ gives a continuous function for $|x| < 1$.