Strictly Monotone functions

A function f is said to be order preserving on $[a, b]$ if, for *all* x and y in this interval, $x < y$ implies that $f(x) < f(y)$. It is said to be order reversing on $[a, b]$ if, for *all* x and y in this interval, $x < y$ implies that $f(x) > f(y)$; equivalently, we require that $-f$ is order preserving. A function is said to be strictly monotone if it is either order preserving or it is order reversing; equivalently, we required that either f or $-f$ is order preserving. Note that the inequality must be the same for *all* x, y in the interval $[a, b]$ with $x < y$.

Strictly monotone continuous functions

Given that f is an order preserving continuous function on the interval $[a, b]$. By the intermediate value theorem, given any x that lies in the interval $[f(a), f(b)]$ there is a y in the interval $[a, b]$ for which $f(y) = x$. Moreover, if $z < y$, then $f(z) < f(y) = x$ and if $z > y$, then $f(z) > f(y) = x$. Hence, the y is *uniquely* determined by x. Thus, we have a *function* g given by $g(x) = y$. Can we say that this function is continuous?

Let $(x_n)_{n \geq 1}$ by an increasing sequence in $[f(a), f(b)]$. Let $y_n = g(x_n)$ with g as above. Since $f(y_n) = x_n$ and f is order preserving, it follows that $(y_n)_{n \geq 1}$ is also an increasing sequence in $[a, b]$. Let y denote the least upper bound of $(y_n)_{n \geq 1}$. By the continuity of f, we have

$$f(y) = f(\lim(f(y_n))_{n \geq 1}) = \lim(f(y_n))_{n \geq 1} = \lim(x_n)_{n \geq 1} = x$$

where x is the least upper bound of $(x_n)_{n \geq 1}$. It follows that $g(x) = y$. In other words, g preserves least upper bounds of increasing sequences. A similar argument will show that g preserves greatest lower bounds of decreasing sequences.

Let $(x_n)_{n \geq 1}$ be any sequence in $[f(a), f(b)]$ and put $y_n = g(x_n)$. Now, consider $c_n = \max\{x_1, \ldots, x_n\}$ and $d_n = \max\{y_1, \ldots, y_n\}$. Since we are taking the maximum element of a finite set $c_n = x_p$ for some p in the range $1, \ldots, n$; similarly $d_n = y_q$ for some q in the range $1, \ldots, n$. Since f is order preserving we see that $p = q$ and $c_n = x_p = f(y_p) = f(d_n)$. The sequences $(c_n)_{n \geq 1}$ and $(d_n)_{n \geq 1}$ are increasing sequences. By the previous paragraph we see that if c is the least upper bound of $(c_n)_{n \geq 1}$ and d is the least upper bound of $(d_n)_{n \geq 1}$, then $f(d) = c$. By definition of sup, we have $c = \sup(x_n)_{n \geq 1}$ and $d = \sup(y_n)_{n \geq 1}$. Hence, we have shown that g preserves the supremum of sequences. A similar argument will show that g preserves the infimum of sequences.

Finally, from the definition of limit superior and limit inferior in terms of supremum and infimum we see that g preserves limit superior and limit inferior. It follows that g preserves limits and so g is continuous.

A similar argument can be made for order reversing functions f or, more simply the above argument can be applied to $-f$ and the fact that multiplication by -1 is continuous can be used.

In summary, we have shown

> If f is a strictly monotone continuous function then it has an inverse function g which is also continuous (and is also strictly monotone).

Examples

The above reasoning can be applied to the function $f(x) = x^n$ on any interval of the form $[0, a]$ where it is order preserving. This shows that there is a continuous function g on the interval $[0, a^n]$ so that $(g(x))^n = x$ for all x in this interval. Since a can be any positive number, we see that this function g is actually defined for all positive x. This is the n-th root function for non-negative numbers and is usually denoted by $x^{(1/n)}$ or $\sqrt[n]{x}$.

The above reasoning can also be applied to the function $\exp(x)$ on any interval of the form $[0, a]$ where it is obviously order preserving. (In fact, it is order preserving on *any* interval as we shall see in the assignment.) This shows that there is a continuous function g on the interval $[\exp(0), \exp(a)]$ so that $\exp(g(x)) = x$ for any x in this interval. Since a can be any positive number, we see that this function g is defined for all positive $x \geq 1 = \exp(0)$. This is the "logarithm" function and is usually denoted by log. (In some non-mathematics books you may see the notation $\log_e$ or ln used for this function, but in this course we will stick to this meaning for log.) We will see a different approach to this function later.