Power Series and Uniform convergence

(2 marks) 1. Consider the functions $f_n(x) = 2^{2n}x^n(1-x)^n$. Is the sequence (f_n) uniformly convergent in $[0, 1]$?

2. Show that each of the following the series gives a continuous function in $|x| < 1$.

(1 mark) (a) $\sum_{n=0}^{\infty} \frac{1}{n+1}x^n$.

(1 mark) (b) $\sum_{n=1}^{\infty} \frac{n}{1+n}x^n$.

(1 mark) 3. Show that the series $\sum_{n=1}^{\infty} nx^n$ gives a continuous function in $|x| < 1$.

4. For any number α and a positive integer k, we define the function

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha - 1) \cdots (\alpha - k + 1)}{k!}$$

In the following sequence of exercises, we will show that the series

$$f_\alpha(x) = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

converges to a continuous function for $|x| < 1$. This power series is called the (generalised) "binomial series" for α-th power of $(1 + x)$.

(1 mark) (a) Show that if $-1 \leq \alpha < 0$, then for all positive integers k we have

$$\left|\binom{\alpha}{k}\right| \leq 1$$

(1 mark) (b) Assuming $\alpha \geq 0$ choose a positive integer r so that $-1 \leq \alpha - r < 0$ (by Archimedean principle!). Then for any positive integer k show that

$$\left|\binom{\alpha}{k}\right| \leq C_r$$

for some constant C_r depending only on r (and not on k).

(1 mark) (c) Assume that there is a positive integer r so that $-r \leq \alpha < -r + 1$ and show that

$$\left|\binom{\alpha}{k}\right| \leq \binom{r + k - 1}{r - 1}$$

(1 mark) (d) For a fixed r show that $\binom{r+k-1}{r-1}$ is a polynomial function of k of degree $r - 1$.

(2 marks) (e) In each case above, use the results already proved in the notes to conclude that the power series $f_\alpha(x)$ converges to define a continuous function in $|x| < 1$.