Solutions to Assignment 6

1. For each of the following functions defined in the interval $(-1, 1)$, find an n such that $|f_p(x) - f_p(y)| < 1/5$ whenever x and y lie in this interval and satisfy $|x - y| < 1/n$.

(1 mark) (a) $f_1(x) = x + 2$

Solution: We note that $f(x) - f(y) = x - y$. So $n = 5$ is adequate.

(1 mark) (b) $f_2(x) = x^2 + 2$

Solution: We note that $f_2(x) - f_2(y) = (x - y)(x + y)$. Note that $|x + y| \leq 2$ for x and y in $(-1, 1)$. So $|f_2(x) - f_2(y)| \leq 2|x - y|$. So $n = 10$ is adequate.

(1 mark) (c) $f_3(x) = 1/(x^2 + 2)$

Solution: Since $f_3(x) = 1/f_2(x)$, we note that

$$f_3(x) - f_3(y) = \frac{f_2(y) - f_2(x)}{f_2(x)f_2(y)}$$

Since $|f_2(z)| \geq 2$ for z in $(-1, 1)$, we see that $|f_3(x) - f_3(y)| \leq (1/2)|f_2(x) - f_2(y)|$ in this interval. So $n = 5$ is adequate.

(1 mark) (d) $f_4(x) = (x + 2)/(x^2 + 2)$

Solution: Since $f_4(x) = f_3(x) \cdot f_1(x)$, we note that

$$f_4(x) - f_4(y) = f_1(x) \cdot (f_3(x) - f_3(y)) + f_3(y) \cdot (f_1(x) - f_1(y))$$

Since $|f_1(z)| \leq 3$ for z in $(-1, 1)$, and $|f_3(z)| \leq 1/2$ for z in $(-1, 1)$, we see that

$$|f_4(x) - f_4(y)| \leq 3|f_3(x) - f_3(y)| + (1/2)|f_1(x) - f_1(y)|$$

in this interval.So $n = 30$ is adequate.

2. We are given a function f on $[0, 1]$ so that for any x, y in this interval, we have $|f(x) - f(y)| < 3|x - y|$.

(1 mark) (a) Show that f is continuous in the interval $[0, 1]$.

Solution: Given $\epsilon > 0$ we choose $\delta = \epsilon/3$. Due to the given inequality, we get, for any x, y in this interval, that if $|x - y| < \delta$, then

$$|f(x) - f(y)| < 3|x - y| < 3\delta = \epsilon$$

(1 mark) (b) At what points of $[0, 1]$ will you compute f so that you get a table of values which you can use to get the approximate value of f up to an accuracy of $1/20$.

> **Solution:** For $\epsilon = 1/20$, we need $\delta = 1/60$. So, we can compute $f(p/60)$ for $p = 0, 1, \ldots, 59$. For any x in the interval $[0, 1]$, we take p to be the largest integer less than $60x$ and put $f(p/60)$ as the approximate value for $f(x)$.

3. Suppose that f and g are given as continuous functions in some interval I of the number line.

(1 mark) (a) Show that the function $|f|$ defined by

$$|f|(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

is a continuous function on I. (Hint: Use composition of continuous functions.)

> **Solution:** The function $a(x) = |x|$ was shown to be a continuous function in the notes. It follows that $a \circ f$ is a continuous function. We note that $|f| = a \circ f$.

(1 mark) (b) Define the functions f_+ and f_- by

$$f_+ = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

and

$$f_- = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Show that $|f| = f_+ + f_-$ and $f = f_+ - f_-$.

> **Solution:** We see that, at all points x where $f(x) \geq 0$, we have $f_- = 0$ and so $|f|(x) = f(x) = f_+(x)$ as required. Similarly, at all points x where $f(x) < 0$, we have $f_+ = 0$ and so $|f|(x) = -f(x) = f_-(x)$ as required.

(1 mark) (c) Use the previous two parts to show that f_+ and f_- are continuous functions on I.

> **Solution:** We add and subtract the above two identities to obtain
>
> $$f_+ = \frac{|f| + f}{2} \text{ and } f_+ = \frac{|f| - f}{2}$$
>
> By the arithmetic properties of continuous functions, it follows that f_+ and f_- are continuous functions.

(1 mark) (d) Use $h = f - g$ to denote the difference of the two functions and show that for all x in I,

$$g(x) + h_+(x) = \max\{f(x), g(x)\}$$

Solution: At all points x where $f(x) \geq g(x)$, we have $h_+(x) = f(x) - g(x)$ and so $g(x) + h_+(x) = f(x)$ at these points. On the other hand, at points x where $f(x) < g(x)$, we have $h_+(x) = 0$ and so $g(x) + h_+(x) = g(x)$.

(1 mark) (e) Show that the function $\max\{f, g\}$ defined by

$$(\max\{f, g\})(x) = \max\{f(x), g(x)\}$$

is a continuous function in I. Similarly, for $\min\{f, g\}$.

Solution: By the arithmetic of continuous functions h is a continuous function. As seen above, this means h_+ is a continuous function. Again applying the arithmetic of continuous functions, we see that $g + h_+$ is also continuous. By the previous exercise, this is the same as $\max\{f, g\}$.

We note that $\min\{f, g\} = -\max\{-f, -g\}$, so by the arithmetic of continuous functions and the previous paragraph, this is a continuous function. A different proof can be given by equating it to $g - h_-$.