Limits of functions

Given any positive number x, we have seen that the sequence $((1 + x/n)^n)_{n\geq 1}$ is a bounded increasing sequence. If $f(x)$ denotes the least upper bound of this sequence, is this f a continuous function of x?

Similarly, given any number $x > 1$, we have seen that the sequence defined by $z_1 = 1$ and $z_{n+1} = (xz_n + x)/(z_n + x)$ results in a bounded increasing sequence whose limit is a number $f(x)$ such that $f(x)^2 = x$. Is this a continuous function of x?

In each of the above cases, the n-th term of the sequence is a function of x. So we need to understand the meaning of convergence of a sequence of functions.

Pitfall

For each positive integer n, let $f_n(x) = x^n$. This is a continuous function on the interval $[0, 1]$. Moreover, for each fixed x, the sequence $(f_n(x))_{n\geq 1}$ converges. In fact, if $x < 1$, then, as we have seen earlier, this sequence converges to 0. When $x = 1$, this is a constant sequence with term 1 and so it converges to 1. Thus, the limiting function is defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

It is obvious that this is *not* a continuous function. (The sequence $((1 - (1/n)))$ converges to 1 but the values $f(1 - (1/n))$ are all 0, while $f(1) = 1$.)

In other words, if $(f_n)_{n\geq 1}$ is a sequence of continuous functions so that $(f_n(x))$ converges for all values of x in some region, this alone *does not* ensure that the "limiting function" is a continuous function. We need something more.

A positive example

For each positive integer n we define the function $e_n(x)$ by the formula

$$e_n(x) = 1 + x + \cdots + \frac{x^n}{n!} = \sum_{k=1}^{n} \frac{x^k}{k!}$$

For x lying in the interval $[0, 1]$, the sequence $(e_n(x))_{n\geq 1}$ is an increasing sequence. We have shown earlier that it is bounded. Hence, it has a limit (which also is its least upper bound) which we can denote as $e(x)$. The question we ask ourselves is whether this is a continuous function. To see this, we note that

$$e(x) - e(y) = (e(x) - e_n(x)) + (e_n(x) - e_n(y)) + (e_n(y) - e(x))$$

Since, e_n is continuous, we can make the second term as small as we want by making x and y close enough. The problem is to make $e(z) - e_n(z)$ *uniformly* small for *all* z in $[0, 1]$ by choosing n large enough.

Now, $e(x) - e_n(x)$ is given by the sum of the series $\sum_{k=n+1}^{\infty}(x^k/k!)$ which is bounded above by $\sum_{k=n+1}^{\infty}(1/k!)$ for x in $[0,1]$. The latter series is in turn bounded above

$$\sum_{k=n+1}^{\infty} \frac{1}{k!} = \sum_{p=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{(n+1)\cdots(n+p)} \leq \frac{1}{n!} \cdot \sum_{p=1}^{\infty} \frac{1}{(n+1)^p} = \frac{1}{n!} \cdot \frac{1}{n}$$

by comparing with the geometric series.

Now, given $\epsilon > 0$, we can choose n so that $e(x) - e_n(x) < \epsilon/3$ for *all* x in $[0,1]$. For *that* choice of n, by the continuity of e_n, we can choose δ so that $|e_n(x) - e_n(y)| < \epsilon/3$ for $|x - y| < \delta$. It then follows that

$$|e(x) - e(y)| \leq |e(x) - e_n(x)| + |e_n(x) - e_n(y)| + |e_n(y) - e(x)| < \epsilon$$

This proves continuity of $e(x)$. It is not too difficult to extend this argument to *any* interval $[a,b]$ in the real line. Hence, we see that the series $\sum_{k=0}^{\infty}(x^k/k!)$ defines a continuous function for *all* x. This function is called the exponential function and denoted as $\exp(x)$. We will study its properties later.

Uniform convergence

When we examine the above proof of continuity, we note that the *key* point was that given any $\epsilon > 0$ we can choose an n so that $|e(x) - e_n(x)| < \epsilon$ for *all* x in the interval $[0,1]$.

We say that a sequence $(f_n)_{n\geq 1}$ of functions converges *uniformly* to a function f in an interval $[a,b]$ if, given a positive integer k, there is an n_k so that $|f_n(x) - f(x)| < 1/k$ for all $n \geq n_k$ *and* for all x in the interval $[a,b]$.

The *uniform* choice of n for all x in the interval $[a,b]$ is the crucial point for us.

Now, suppose that f_n are all continuous in the interval $[a,b]$, we then claim that f is *also* continuous in this interval. The proof is essentially the same as the proof given above for the sequence $(e_n)_{n\geq 1}$. Given a positive integer k, we pick an n so that $n > n_{3k}$ so that $|f(x) - f_n(x)| < 1/(3k)$ for *all* x in $[a,b]$. For *this* choice of n, we choose some p so that $|f_n(x) - f_n(y)| < 1/(3k)$ for all x, y in the interval $[a,b]$ with $|x - y| < 1/p$. We then have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(x)| < 1/k$$

as required.

As in the case of convergence of numbers it is sometimes convenient to talk about uniform convergence of functions *without* knowing the limiting function.

We say that a sequence $(f_n)_{n\geq 1}$ is "uniformly convergent" in an interval $[a,b]$ if, given a positive integer k, there is an n_k so that $|f_p(x) - f_q(x)| < 1/k$ for all $p, q \geq n_k$ and for all x in the interval $[a,b]$.

Note that, for a fixed x this implies that $(f_n(x))_{n\geq 1}$ satisfies Cauchy's criterion for convergence. Hence, it converges to a limit which we can call $f(x)$. This defines a function f on $[a, b]$. Now, given k, we note that if $p \geq n_{k+1}$, then

$$|f(x) - f_p(x)| = \lim(|f_n(x) - f_p(x)|)_{n\geq 1} \leq 1/(k+1) < 1/k$$

Hence, the sequence of functions $(f_n)_{n\geq 1}$ converges uniformly to f.

In particular, we note that if $(f_n)_{n\geq 1}$ is a uniformly convergent sequence of continuous functions on the interval $[a, b]$, then the limit function is defined on $[a, b]$ and is a continuous function on this interval.

Piecewise linear approximation

We would like to find a "good" approximation of a continuous function f on the interval $[a, b]$.

Given an error bar $1/k$ (typically $k = 10^r$ for $(r-1)$ places of decimal), we have seen that there is an n_k so that if $|x - y| < 1/n_k$ and x, y lie in $[a, b]$ we have $|f(x) - f(y)|1/k$.

Thus, it is enough to calculate $f(a+p/n_k)$ for values of p from 0 up to $(b-a)/n_k$ (or the largest multiple of $1/n_k$ less than it). This allows us to create a table of values which can be used in place of evaluation of f for each given x. In other words, given x in $[a, b]$ we can take p to be the greatest integer less than (or equal to) $(x - a)n_k$; then $f(a + p/n_k)$ is an approximation of $f(x)$ with error at most $1/k$.

We may not be satisfied with this way of doing the approximation since these approximate values give a "step" function that jumps at $a+p/n_k$ for each integer p.

A different approach is to define a new function f_k as follows. Let r be the integer part of $(b-a)n_k$ and use the notation $a_p = a+p/n_k$ for $p = 0, \ldots, r$ and $a_{r+1} = b$. For x lying in the interval $[a_p, a_{p+1}]$ we put $t = (x - a_p)/(a_{p+1} - a_p)$ and note that $x = (1 - t) \cdot a_p + t \cdot a_{p+1}$. We then put $f_k(x) = (1 - t)f(a_p) + tf(a_{p+1})$. It is clear that this function f_k can easily be implemented as a program by giving the table of values $f(a_p)$.

We note that this function satisfies $f_k(a_p) = f(a_p)$. For this reason, we say that it "interpolates" the (values of the) function f at these points or that f_k is a piecewise linear interpolation of f.

As proved in the previous section, this function f_k is continuous. Moreover, we notice that for x in $[a_p, a_{p+1}]$, we have

$$\begin{aligned}|f(x) - f_k(x)| &= |f(x) - ((1 - t)f(a_p) + tf(a_{p+1}))| \\ &\leq (1 - t)|f(x) - f(a_p)| + t|f(x) - f(a_{p+1})|\end{aligned}$$

Since $a_{p+1} - a_p \leq 1/n_k$ and x is "trapped" in between, we see that $|f(x) - f(a_p)| < 1/k$ and $|f(x) - f(a_{p+1})| < 1/k$. So, it follows that

$$\begin{aligned} |f(x) - f_k(x)| &\leq (1-t)|f(x) - f(a_p)| + t|f(x) - f(a_{p+1})| \\ &< (1-t)(1/k) + t(1/k) = 1/k \end{aligned}$$

In other words, f_k is *uniformly* close to f with error at most $1/k$.

In summary, we have shown that there is a sequence $(f_k)_{k \geq 1}$ of piecewise linear continuous functions that converges uniformly to a given continuous function f on an interval $[a, b]$.

Such an approximation is very useful when computing f at every point is very difficult or computationally intensive and a good enough approximation will do.