Continuous Functions

1. For each of the following functions defined in the interval $(-1, 1)$, find an n such that $|f_p(x) - f_p(y)| < 1/5$ whenever x and y lie in this interval and satisfy $|x - y| < 1/n$.

(1 mark) (a) $f_1(x) = x + 2$

(1 mark) (b) $f_2(x) = x^2 + 2$

(1 mark) (c) $f_3(x) = 1/(x^2 + 2)$

(1 mark) (d) $f_4(x) = (x + 2)/(x^2 + 2)$

2. We are given a function f on $[0, 1]$ so that for any x, y in this interval, we have $|f(x) - f(y)| < 3|x - y|$.

(1 mark) (a) Show that f is continuous in the interval $[0, 1]$.

(1 mark) (b) At what points of $[0, 1]$ will you compute f so that you get a table of values which you can use to get the approximate value of f up to an accuracy of $1/20$.

3. Suppose that f and g are given as continuous functions in some interval I of the number line.

(1 mark) (a) Show that the function $|f|$ defined by

$$|f|(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

is a continuous function on I. (Hint: Use composition of continuous functions.)

(1 mark) (b) Define the functions f_+ and f_- by

$$f_+ = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

and

$$f_- = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Show that $|f| = f_+ + f_-$ and $f = f_+ - f_-$.

(1 mark) (c) Use the previous two parts to show that f_+ and f_- are continuous functions on I.

(1 mark) (d) Use $h = f - g$ to denote the difference of the two functions and show that for all x in I,

$$g(x) + h_+(x) = \max\{f(x), g(x)\}$$

(1 mark) (e) Show that the function $\max\{f, g\}$ defined by

$$(\max\{f, g\})(x) = \max\{f(x), g(x)\}$$

is a continuous function in I. Similarly, for $\min\{f, g\}$.