Solutions to Assignment 5

(3 marks) 1. Consider the following sequences

(a) The sequence $(a_n)_{n\geq 1}$ with

$$a_n = n^3 \cdot (3/4) - 25n^2$$

(b) The sequence $(b_n)_{n\geq 1}$ with

$$a_n = n^3 \cdot (3/5) + 50 \cdot n^2 + 10n$$

(c) The sequence $(c_n)_{n\geq 1}$ with

$$c_n = \frac{3^n - 100 \cdot n^4}{2^n \cdot 500}$$

Which sequence dominates which sequence for large values of n?

Solution: As seen in the notes, $(1+x)^n$ dominates any polynomial function of n for any $x > 0$. Secondly,

$$n^3(3/4) - n^3(3/5) + p \cdot n^2 + q \cdot n + r = n^3(3/20) + p \cdot n^2 + q \cdot n + r$$

Since n^3 dominates any $pn^2 + qn + r$ for any fixed p, q and r, we see that the order (for n sufficiently large is):

$$c_n > a_n > b_n$$

2. Which of the following series of positive terms converges. Justify your answer by giving an upper bound for the sum.

(1 mark) (a) The series $\sum_{k=1}^{\infty}(6/7)^k$

Solution: By the sum of the geometric series, we have

$$\sum_{k=1}^{n}(6/7)^k = \frac{6}{7} \cdot \frac{1-(6/7)^{k+1}}{1-(6/7)} = 6\left(1-(6/7)^{k+1}\right) < 6$$

(1 mark) (b) The series $\sum_{k=0}^{\infty} 1/(2k+1)$

Solution: We have $1/(2k+1) > 1/(2k+2)$. By the examination of the harmonic series in the notes, we have seen that

$$\sum_{k=0}^{2^n-1}(1/2k+2) = (1/2)\sum_{k=0}^{2^n-1} 1/(k+1) = (1/2)\sum_{k=1}^{2^n}(1/k) \geq (1/2)\left(1+(1/2)\,n\right)$$

Thus, this sequence is unbounded.

(1 mark) (c) The series $\sum_{k=2}^{\infty} 1/(k^2 - 1)$

> **Solution:** We note $1/((k+1)^2 - 1) < 1/k^2$. By the examination of the square harmonic series in the notes, we have seen that
>
> $$\sum_{k=1}^{2^n-1}(1/k^2) = \sum_{k=1}^{2^n-1}(1/k^2) < \sum_{k=0}^{n}(1/2^k) < 2$$

(3 marks) 3. Show that the following sequence is *increasing* and bounded above and below: $x_1 = 1$ and

$$x_{n+1} = \frac{13x_n + 13}{x_n + 13}$$

(Hint: Compare x_n^2 with 13.)

> **Solution:** We note that $x_1^2 < 13$. We wish to show by induction that $x_n^2 < 13$. *Assuming* this for a given n, we calculate
>
> $$x_{n+1}^2 - 13 = \frac{(13x_n + 13)^2 - 13(x_n + 13)^2}{(x_n + 13)^2} = \frac{156(x_n^2 - 13)}{(x_n + 13)^2} < 0$$
>
> This shows, by the principle of induction, that $x_n^2 < 13$ for all n.
>
> If $x_n \geq 4$ for some n, then $x_n^2 \geq 16$. Since $x_n^2 < 13$ for all n, this means that $x_n < 4$ for all n.
>
> We note that $x_1 > 0$. Assuming that $x_n > 0$ for some n, the formula
>
> $$x_{n+1} = \frac{13x_n + 13}{x_n + 13}$$
>
> shows that $x_{n+1} > 0$. By induction, we see that $x_n > 0$ for all n.
>
> We calculate
>
> $$x_n - \frac{13x_n + 13}{x_n + 13} = \frac{x_n^2 - 13}{x_n + 13} < 0$$
>
> It follows that $x_n < x_{n+1}$ for all n.

4. Define the sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, $(z_n)_{n\geq 1}$ as follows. First of all we have the identity

$$z_n = \frac{2x_n + 3y_n}{5}$$

We define $x_1 = 2$ and $y_1 = 0$. Finally, x_n and y_n are defined defined as:

1. If $z_n^3 \leq 5$, then $y_{n+1} = z_n$ and $x_{n+1} = x_n$.

2. If $z_n^3 > 5$, then $y_{n+1} = y_n$ and $x_{n+1} = z_n$.

(1 mark) (a) Show that $(2a + 3b)/5$ lies between a and b. (Correction: Original question had $(a + b)/2$.)

> **Solution:** If $a \geq b$ then
>
> $$a - \frac{2a+3b}{5} = \frac{3(a-b)}{5} \geq 0$$
>
> and
>
> $$b - \frac{a+b}{2} = \frac{2(b-a)}{5} \leq 0$$
>
> Similar argument in the case $a \leq b$.

(1 mark) (b) Show by induction that $x_n > z_n > y_n$ for all n.

> **Solution:** We have $x_1 > y_1$, hence $x_1 > z_1 > y_1$. We will and prove by induction that $x_n > z_n > y_n$. Assume this for some n.
>
> If $z_n^3 \leq 5$, then $y_{n+1} = z_n$ and $x_{n+1} = x_n$. So we have $x_{n+1} > y_{n+1}$. Using the previous part we get $x_{n+1} > z_{n+1} > y_{n+1}$ as required.
>
> If $z_n^3 < 5$, then $y_{n+1} = y_n$ and $x_{n+1} = z_n$. So we have $x_{n+1} > y_{n+1}$. Using the previous part we get $x_{n+1} > z_{n+1} > y_{n+1}$ as required.

(1 mark) (c) Show that $(x_n)_{n\geq 1}$ is an decreasing sequence and $(y_n)_{n\geq 1}$ is a increasing sequence. (Correction: The original question had "increasing" and "decreasing" interchanged!)

> **Solution:** Since at every stage, either $x_{n+1} = x_n$ or $x_{n+1} = z_n < x_n$, we see that $(x_n)_{n\geq 1}$ is a decreasing sequence. Similarly $(y_n)_{n\geq 1}$ is an increasing sequence.

(1 mark) (d) Show by induction that $|x_n - y_n| \leq 2 \cdot (3/5)^{n-1}$. (Correction: The original question had $1/2^{n-2}$.)

> **Solution:** We note that $x_1 - y_1 = 2$. We also note that $z_1 - y_1 = 1$ and $x_1 - z_1 = 1$. By induction, we claim that
>
> $$x_n - y_n \leq 2 \cdot (3/5)^{n-1} \; ; \; z_n - y_n \leq 2 \cdot (3/5)^n \; ; \; x_n - z_n \leq 2 \cdot (3/5)^n \; ;$$
>
> Let us assume this for a given n. We note that in the case(2) above, we have
>
> $$(x_{n+1}, y_{n+1}) = (z_n, y_n)$$
>
> In the case(1) above, we have
>
> $$(x_{n+1}, y_{n+1}) = (x_n, z_n)$$

So the first equality is obtained. Now, z_{n+1} is $(2x_{n+1} + 3y_{n+1})/5$. So the next two inequalities follow as a consequence of the calculation in part (1) above.

(1 mark) (e) Show the following inequalities.

$$\limsup(z_n)_{n\geq 1} \geq \sup(y_n)_{n\geq 1}$$
$$\liminf(z_n)_{n\geq 1} \geq \inf(x_n)_{n\geq 1}$$

Solution: We have seen above that $z_n - y_n > 0$. This shows that $z_n \geq y_n$. It follows that $\sup(z_n)_{n\geq k} \geq \sup(y_n)_{n\geq k}$. Since $(y_n)_{n\geq 1}$ is increasing, we see that for all k, we have

$$\sup(y_n)_{n\geq 1} = \sup(y_n)_{n\geq k}$$

We now calculate

$$\limsup(z_n)_{n\geq 1} = \inf(\sup(z_n)_{n\geq k})_{k\geq 1} \geq \inf(\sup(y_n)_{n\geq k})_{k\geq 1} = \sup(y_n)_{n\geq 1})$$

The argument for z_n and x_n is similar.

(1 mark) (f) Show that all three sequences converge and have the same limit.

Solution: We have $x_n - y_n \leq 2 \cdot (3/5)^{n-1}$. Hence,

$$\begin{aligned}\inf(x_n)_{n\geq 1} &= \lim(x_n)_{n\geq 1} \\ &= \limsup(x_n)_{n\geq 1} \leq \limsup(y_n + 2\cdot(3/5)^{n-1})_{n\geq 1} = \limsup(y_n)_{n\geq 1} + 0 \\ &= \limsup(y_n)_{n\geq 1}\end{aligned}$$

Thus,

$$\begin{aligned}&\liminf(z_n)_{n\geq 1} \\ &\leq \inf(x_n)_{n\geq 1} \leq \limsup(y_n)_{n\geq 1} \\ &\leq \limsup(z_n)_{n\geq 1} \leq \liminf(z_n)_{n\geq 1}\end{aligned}$$

Hence, we have equality as required.