Solutions to Assignment 4

1. Find the limit superior and the limit inferior of the following sequences. (You may use limit calculations from the notes.)

(1 mark) (a) $\left(1+\frac{1}{n}\right)_{n\geq 1}$

Solution: We note that $1+1/n$ is a *decreasing* sequence with greatest lower bound 1. Hence, both limit superior and limit inferior are 1.

A different approach is as follows.

We note that $\sup((1+1/n))_{n\geq k} = 1+1/k$. It follows that $\limsup(1+1/n)_{n\geq 1} = \inf(1+1/k)_{k\geq 1} = 1$.

We note that $\inf((1+1/n))_{n\geq k} = 1$. It follows that $\liminf(1+1/n)_{n\geq 1} = \inf(1)_{k\geq 1} = 1$.

(1 mark) (b) $\left((-1)^{n+1}+\frac{(-1)^n}{n}\right)_{n\geq 1}$

Solution: We note that

$$x_n = (-1)^{n+1} + (-1)^n/n) = \begin{cases} 1-1/n & n \text{ odd} \\ -1+1/n & n \text{ even} \end{cases}$$

It follows that the supremum of $(x_n)_{n\geq k}$ is 1. Hence, the limit superior of this sequence is 1.

Similarly, the infimum of $(x_n)_{n\geq k}$ is -1. Hence, the limit inferior of this sequence is -1.

(1 mark) (c) $\left(\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n}\right)\right)_{n\geq 1}$

Solution: We note that

$$x_n = \left(1+\frac{1}{n}\right)^n - 1 - \frac{1}{n}$$

We have already shown that $(1+(1/n))^n$ is increasing with n and is bounded. Let e denote its least upper bound. We have also seen that $-\frac{1}{n}$ is increasing with n with least upper bound 0. From the principle of addition of least upper bounds we see that for *all* k, we have

$$\sup(x_n)_{n\geq k} = e-1-0 = e-1$$

Since the sequence is increasing, we see that

$$\inf(x_n)_{n\geq k} = x_k$$

It follows that $\limsup(x_n)_{n\geq 1} = e-1$ and also

$$\liminf(x_n)_{n\geq 1} = \sup(x_k)_{k\geq 1} = e-1$$

(1 mark) (d) $\dfrac{n-2}{n^2+2n+1}$

Solution: We note that

$$x_n = \frac{1}{n+2+\frac{1}{n}} - \frac{2}{n^2+2n+1}$$

Hence $x_n \leq (1/n)$ and $x_n \geq -(2/n^2)$. It follows that

$$\begin{aligned} \sup(x_n)_{n\geq k} &\leq \sup(1/n)_{n\geq k} &= 1/k \\ \sup(x_n)_{n\geq k} &\geq \sup(-2/n^2)_{n\geq k} &= 0 \end{aligned}$$

We then take infimum of both sides to get

$$\begin{aligned} \limsup(x_n)_{n\geq 1} &\leq \inf(1/k)_{k\geq 1} &= 0 \\ \limsup(x_n)_{n\geq 1} &\geq \inf(0)_{k\geq 1} &= 0 \end{aligned}$$

Thus, we see that $\limsup(x_n)_{n\geq 1} = 0$.
A similar argument shows that $\liminf(x_n)_{n\geq 1} = 0$.

(1 mark) (e) $\dfrac{n^2-(-1)^n 2n-1}{n^2+2n+1}$

Solution: We note that

$$x_n = \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} - (-1)^n\frac{2}{n+2+\frac{1}{n}} - \frac{1}{n^2+2n+1}$$

Hence $x_n \leq 1+(2/n)$. Further, for any t such that $0 < t < 1$, we have $1 > (1+t)(1-t)$ so $1/(1+t) > 1-t$. Now, for $n > 3$, we have $0 < (2/n)+(1/n^2) < 1$ so

$$\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} \geq 1 - \frac{2}{n} - \frac{1}{n^2}$$

So, for $n \geq 3$, we get

$$x_n \geq 1 - \frac{4}{n} - \frac{2}{n^2}$$

It follows that for $k > 3$,

$$\begin{aligned} \sup(x_n)_{n\geq k} &\leq \sup(1+(2/n))_{n\geq k} &= 1+2/k \\ \sup(x_n)_{n\geq k} &\geq \sup(1-(4/n)-(2/n^2))_{n\geq k} &= 1 \end{aligned}$$

We then take infimum of both sides to get

$$\begin{aligned} \limsup(x_n)_{n\geq 1} &\leq \inf(1+2/k)_{k\geq 1} &= 1 \\ \limsup(x_n)_{n\geq 1} &\geq \inf(1)_{k\geq 1} &= 1 \end{aligned}$$

Thus, we see that $\limsup(x_n)_{n\geq 1} = 1$.
A similar argument shows that $\liminf(x_n)_{n\geq 1} = 1$.

(1 (bonus)) (f) For each n, let k_n be such that 2^{k_n} is the *smallest* power of 2 which is greater than n; in other words $2^{k_n-1} \leq n < 2^{k_n}$. Now take the sequence $(n/2^{k_n})_{n\geq 1}$.

(1 (bonus)) (g) $(\sin(n))_{n\geq 1}$.

2. Show that the following sequences have a limit.

(1 mark) (a) The sequence $(x_n)_{n\geq 1}$ where

$$x_n = 1 + \frac{2}{n} + \frac{3}{n^2}$$

Solution: The sequence $(1/n)_{n\geq 1}$ has a limit (which is 0). It follows that $(1/n^2 = (1/n)^2)_{n\geq 1}$ has a limit (which is also 0!). Since limits are preserved under addition and multiplication, we see that

$$\lim(x_n)_{n\geq 1} = \lim(1)_{n\geq 1} + \lim(2/n)_{n\geq 1} + \lim(3/n^2)_{n\geq 1}$$

Hence, the limit exists (and is equal to 1).

(1 mark) (b) The sequence $(x_n)_{n\geq 1}$ where

$$x_n = \frac{1 + 2\frac{1}{n} + 3\frac{1}{n^2}}{1 - 2\frac{1}{n} + 3\frac{1}{n^2} - 4\frac{1}{n^3}}$$

Solution: The sequence $(1/n)_{n\geq 1}$ has a limit (which is 0). It follows that $(1/n^2 = (1/n)^2)_{n\geq 1}$ has a limit (which is also 0!). Similarly, the sequence $(1/n^3 = (1/n)(1/n^2))_{n\geq 1}$ has a limit. We define

$$\begin{aligned} a_n &= 1 + 2\frac{1}{n} + 3\frac{1}{n^2} \\ b_n &= 1 - 2\frac{1}{n} + 3\frac{1}{n^2} - 4\frac{1}{n^3} \end{aligned}$$

Since limits are preserved under addition, subtraction and multiplication, we see that

$$\lim(a_n)_{n\geq 1} = \lim(1)_{n\geq 1} + \lim(2/n)_{n\geq 1} + \lim(3/n^2)_{n\geq 1}$$
$$\lim(b_n)_{n\geq 1} = \lim(1)_{n\geq 1} - \lim(2/n)_{n\geq 1} + \lim(3/n^2)_{n\geq 1} - \lim(4/n^3)_{n\geq 1}$$

Moreover, the limit of b_n is 1 which is positive. Hence, there is an n_0 such that b_n is positive for $n \geq n_0$. For such n we know that the limit is preserved under division. Hence the limit of (x_n) exists and is given by the formula below.

$$\lim(x_n)_{n\geq n_0} = \frac{\lim(a_n)_{n\geq n_0}}{\lim(b_n)_{n\geq n_0}}$$

(1 mark) (c) The sequence $(x_n)_{n\geq 1}$ where

$$x_n = 1 + 2\left(5 + \frac{1}{n}\right) + 3\left(5 + \frac{1}{n}\right)^2$$

Solution: The sequence $(1/n)_{n\geq 1}$ has a limit. It follows that $(5 + 1/n)_{n\geq 1}$ has a limit. Hence, the sequence $((5 + 1/n)^2)_{n\geq 1}$ has a limit. By the addition of limits, we obtain

$$\lim(x_n)_{n\geq 1} = 1 + 2\lim((5 + 1/n))_{n\geq 1} + 3\lim((5 + 1/n)^2)_{n\geq 1} +$$

Hence the limit of (x_n) exists and is equal to $1 + 2 \cdot 5 + 3 \cdot 5^2 = 86$.

(1 mark) (d) The sequence $(x_n)_{n\geq 1}$ where

$$x_n = \frac{1}{1 + 2\left(5 + \frac{1}{n}\right) + 3\left(5 + \frac{1}{n}\right)^2}$$

Solution: As seen above, the limit of the denominator exists and is *not zero*. By the division of limits for positive limits, we see that (x_n) has a limit.

(1 mark) (e) $\left(\left(1 + \frac{1}{n+1}\right)^n\right)_{n\geq 1}$

Solution: We note that

$$\left(1 + \frac{1}{n+1}\right)^n = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n+1}\right)}$$

Now the numerator is an increasing bounded sequence. Hence, it has a least upper bound which is also its limit. The denominator is a decreasing bound sequence wih greatest lower bound 1 which is also its limit. Thus, by the preservation of limits under division by a sequence with a positive limit, we see that the above sequence also has a limit.