Arithmetic and Least Upper Bound

In order to calculate with and use the least upper bound "construction" in our number system, we need to work out its arithmetic properties.

Before going ahead, let us mention that, given a bounded (above) increasing sequence $(x_n)_{n\geq 1}$, we see that $(-x_n)_{n\geq 1}$ is a bounded (below) decreasing sequence. Thus, all our assertions about properties of the least upper bound can be easily transposed to properties of the greatest lower bound.

Simplest sequence

Given an increasing sequence $x_1, x_2, x_3, \ldots$ of positive numbers, the sequence $1/x_1, 1/x_2, 1/x_3, \ldots$ is decreasing. Hence the sequence

$$1 - \frac{1}{1}, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots$$

is an increasing sequence bounded. Moreover, it is above by 1. How does one prove that its least upper bound is one? This may *seem* obvious, but it always a good idea to test our ability to prove things when we already *know* the answer!

We need to show that if $x < 1$, then x is *not* an upper bound of the above sequence. In other words, we want to show that there is an n so that $x < 1 - 1/n$. By simple arithmetic, this inequality is the same as

$$n > \frac{1}{1-x}$$

Now, we see another use for the Archimedean principle! $1 - x$ is a positive number, so $1/(1-x)$ is also a positive number. By the Archimedean principle, there is a natural number n so that $n = n \cdot 1 > 1/(1-x)$.

Hence, we have proved that the least upper bound of the above sequence is 1.

Adding a constant

Given a bounded increasing sequence $(x_n)_{n\geq 1}$ with least upper bound x. We easily guess that, for any number a, the sequence

$$a + x_1, a + x_2, a + x_3, \ldots$$

is a bounded increasing sequence with least upper bound $a + x$. How does one prove this? First of all, since $x_n < x_{n+1}$ is given, we see that $a + x_n < a + x_{n+1}$, since order is preserved under addition of the same number to both sides. Similarly, $x_n \leq x$ is given so $a + x_n < a + x$ means that this sequence is also bounded above by $a + x$. It follows that the above sequence has a least upper bound y and so $a + x \geq y$. This means that $a + x_n \leq y$ and so $x_n \leq y - a$. It follows that $y - a$ is an upper bound for x_n. Since x is the least upper bound of

$(x+n)_{n\geq 1}$, it follows that $y-a \geq x$. Combining the two statements $y \geq a+x$ and $a+x \geq y$ we obtain $y = a+x$.

As a result of this argument, one can always "translate" a problem of studying the least upper bound of a sequence and get a sequence of positive terms.

Multiplying a positive constant

Given a bounded increasing sequence $(x_n)_{n\geq 1}$ with least upper bound x. We easily guess that, for any positive number $a > 0$, the sequence

$$a \cdot x_1, a \cdot x_2, a \cdot x_3, \ldots$$

is a bounded increasing sequence with least upper bound $a \cdot x$. How does one prove this? First of all, since $x_n < x_{n+1}$ is given, we see that $a \cdot x_n < a \cdot x_{n+1}$, since order is preserved under multiplication of both sides by the same positive number. Similarly, $x_n \leq x$ is given so $a \cdot x_n < a \cdot x$ means that this sequence is also bounded above by $a \cdot x$. It follows that the above sequence has a least upper bound y and so $a \cdot x \geq y$. This means that $a \cdot x_n \leq y$ and so $x_n \leq y/a$. It follows that y/a is an upper bound for x_n. Since x is the least upper bound of $(x+n)_{n\geq 1}$, it follows that $y/a \geq x$. Combining the two statements $y \geq a \cdot x$ and $a \cdot x \geq y$ we obtain $y = a \cdot x$.

(It is worth noting how the above argument merely involved changing addition by multiplication and subtraction by division. Such a "mutation" of the proof that proves something else is called *mutatis mutandis*.)

Adding two sequences

Given bounded increasing sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ with least upper bounds x and y, respectively. We easily guess that the sequence $(x_n + y_n)_{n\geq 1}$ should have least upper bound $x+y$. As before, we easily show that $(x_n+y_n)_{n\geq 1}$ is an increasing sequence and $x_n + y_n \leq x + y_n \leq x + y$. So we need to show that if $z < x+y$, then z is *not* an upper bound. In other words, that there is an n so that $x_n + y_n > z$. Now, $z - y < x$, and x is the least upper bound of (x_n), so there *is* a p so that $z - y < x_p$. Now, $z - x_p < y$, and y is the least upper bound of (y_n), so there is a q so that $z - x_p < y_q$. So we get $z < x_p + y_q$, for some p and q. Let $r \geq \max\{p, q\}$; then $x_p \leq x_r$ and $y_q \leq y_r$, since both (x_n) and (y_n) are increasing sequences. It follows that $x_p + y_q \leq x_r + y_r$ and so $z < x_r + y_r$. Thus, we have shown that:

> the sum of the least upper bound of two bounded increasing sequences
> is the least upper bound of the term-wise sum of these sequences.

Multiplying two sequences

One has to be a little careful while comparing products since order is only preserved under multiplication by *positive* numbers. Hence, we restrict ourselves to bounded increasing sequences (x_n) and (y_n) where $0 < x_1$ and $0 < y_1$; we assume that x and y are the respective least upper bounds as before. We then guess that $x \cdot y$ is the least upper bound of the product sequence $(x_n \cdot y_n)$. After this the proof goes in a *mutatis mutandis* manner to the proof above.

First of all, we note that $x_n \cdot y_n \leq x \cdot y_n \leq x \cdot y$. So $x \cdot y$ *is* an upper bound for the sequence $(x_n \cdot y_n)$.

For a number $z < x \cdot y$ we want to show that there is an element of the sequence $(x_n \cdot y_n)$ which is larger than z. Since y is positive, we also have $(z/y) < x$. Since x is the least upper bound of (x_n), there is a p such that $(z/y) < x_p$. Now, x_p is also positive, so $z < x_p \cdot y$ means that $(z/x_p) < y$. Since y is the least upper bound of (y_n), there is a q such that $(z/x_p) < y_q$. It follows that $z < x_p \cdot y_q$. Now, take $r \geq \max\{p, q\}$ and note that $x_p \cdot y_q \leq x_r \cdot y_r$, since both sequences are increasing. It follows that $z < x_r \cdot y_r$ and so z is not an upper bound for $(x_n \cdot y_n)$. In summary,

> the product of the least upper bound of two bounded increasing sequences of positive terms is the least upper bound of the term-wise product of these sequences.

Some more sequences

Given a sequence $f(n)$ which *dominates* n, one notes that $1/n > 1/f(n)$ for $n \geq n_0$. It follows that the greatest lower bound of $(1/f(n))_{n \geq n_0}$ is less than or equal to 0 which is the greatest lower bound of $(1/n)$. On the other hand $f(n) > n$ is positive for $n \geq n_0$. So 0 *is* a lower bound for $(1/f(n))$. It follows that 0 is *equal* to the greatest lower bound for $(1/f(n))_{n \geq n_0}$.

We apply this to $f(n) = a_0 + a_1 n + \cdots + a_k n^k$ where $a_k > 0$. We can also apply this to the case $f(n) = (1 + x)^n$ where $x > 0$.

Now, suppose that $0 < x < 1$. It follows that $1/x > 1$ and so $y = (1/x) - 1 > 0$. We have $x = 1/(1 + y)$. It follows that the sequence (x^n) has greatest lower bound 0. It follows that $(x^{n+1}/(1 - x))$ has greatest lower bound 0. We earlier saw

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}$$

Thus, we see that the sequence with general term $1 + x + \cdots + x^n$ has least upper bound $1/(1 - x)$. In other words, the sum of the series $\sum_{n=0}^{\infty} x^n$ is $1/(1 - x)$ by the *definition* of such a sum as the least upper bound of the above sequence.