Sequences and Series - I

As mentioned earlier, the notion of sequences is a bit outside usual arithmetic. We have seen various ways to understand the growth of sequences which are given by a "formula" for x_n as a function of n. However, our original aim was to understand *bounded* increasing sequences whereas, under the study of growth, we looked at *unbounded* sequences. We now examine how this helps us to understand the notion of more and more accurate measurements of one number *in terms of* other numbers.

Measurement with a unit

Given a length a which is our "unit" of measurement, we want to measure a length b.

Note that both a and b are lengths and thus positive.

By the Archimedean principle, we can find a natural number n so that $n \cdot a > b$. In fact, $m \cdot a > b$ for *all* $m \geq n$. It follows that the collection of natural numbers r so that $r \cdot a \leq b$ is *finite* since $r < n$. Let r_1 be the largest such r.

If $r_1 \cdot a = b$, then the length b is r_1 in units of a. However, we expect that, in general, there will be a "bit left over". This is $b_1 = b - r_1 \cdot a$ which is positive and $b_1 < a$ by choice of r_1.

Again applying the Archimedean principle, there is an natural number n (not the same n as the earlier paragraph!) so that $n \cdot b_1 > a$. *Equivalently*, we have $a/n < b_1$. Among $r < n$ so that $a/r \leq b_1$ choose the smallest (we are again choosing the smallest from a finite set) and call this r_2.

Since $b_1 < a$, we see that $r_2 \geq 2$. Moreover, we have $a/(r_2 - 1) > b_1$. Since $r_2 \geq 2$ we see that $r_2 - 1 \geq r_2/2$. It follows that $2a/r_2 \geq a/(r_2 - 1) > b_1$. Now, $a/r_2 \leq b_1$, so a/r_2 is a measurement of b_1 in terms of a.

It follows $(r_1 + 1/r_2) \cdot a \leq b$ is a "better" measurement of b in terms of a.

If $b_1 = a_2$, then we are done and b is measured as $r_1 + 1/r_2$ in terms of the unit a.

Otherwise, we need to measure $b_2 = b_1 - a/r_2 > 0$ in terms of a. It is now convenient to replace a by $a_2 = a/r_2$. Since $2a/r_2 > b_1$, it follows that $0 < b_2 < a_2$. The situation is similar to what we had with b_1 and a above.

So we can repeat the above procedure to find $r_3 \geq 2$ so that $a_2/r_3 \leq b_2$ and $a_2/(r_3 - 1) > b_2$. Doing the arithmetic, we see that

$$\left(r_1 + \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}\right) \cdot a \leq b$$

Gives the next better measurement of b in terms of a with possible error $b_3 = b_2 - a_2/r_3$. We have $0 \leq b_3 < a_3$ where $a_3 = a_2/r_3$. So, we can continue this procedure.

We see that the sequence

$$r_1, \left(r_1 + \frac{1}{r_2}\right), \left(r_1 + \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}\right), \ldots$$

bounded by b/a. It is clearly increasing, and so it has a least upper bound. In fact, this least upper bound *is* b/a almost by construction.

The above procedure *is* what is usually followed in order to steadily improve our measurement process. Mathematically, it allows us to express any positive number (here b/a) as the least upper bound of a sequence of the form

$$r_1, \left(r_1 + \frac{1}{r_2}\right), \left(r_1 + \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}\right), \ldots$$

The general term of this sequence takes the form

$$r_1 + \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \cdots + \frac{1}{r_2 \cdot r_3 \cdots r_k}$$

where $r_i \geq 2$ for $i \geq 2$. (Note that r_1 could even be 0.)

Representation of some numbers

The above example leads us to consider, given a sequence of positive integers,

$$r_2, r_3, \ldots \text{ all} \geq 2,$$

the resulting sequence of fractions

$$\frac{1}{r_2}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \frac{1}{r_2 \cdot r_3 \cdot r_4}, \ldots$$

Clearly, this is an increasing sequence. Is it bounded? We note that the terms of this sequence are less than the terms of the sequence

$$\frac{1}{2}, \frac{1}{2} + \frac{1}{2 \cdot 2}, \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2}, \ldots$$

The general term of the latter sequence is

$$\frac{1}{2} + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2^k}$$

We note that, for any number x, we have

$$\begin{aligned}(1-x) \cdot (x + x^2 + \cdots + x^k) \\ = (x + x^2 + \cdots + x^k) - (x^2 + x^3 + \cdots + x^{k+1}) \\ = x - x^{k+1}\end{aligned}$$

In our case, by substituting $x = 1/2$ we get the identity

$$\left(1 - \frac{1}{2}\right) \cdot \left(\frac{1}{2} + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2^k}\right) = \frac{1}{2} - \frac{1}{2^{k+1}} < \frac{1}{2}$$

It follows that

$$\frac{1}{2} + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2^k} < 1$$

Hence, the above sequence of fractions

$$\frac{1}{r_2}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \frac{1}{r_2 \cdot r_3 \cdot r_4}, \ldots$$

is an increasing sequence of fractions bounded by 1. Hence, it has a least upper bound α which is at most 1. Conversely, we can use the previous section to show that *any* positive number $\alpha \leq 1$ either has a finite expression of the form

$$\frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \cdots + \frac{1}{r_2 \cdot r_3 \cdots r_n}$$

or is the least upper bound of a sequence of the form

$$\frac{1}{r_2}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3}, \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \frac{1}{r_2 \cdot r_3 \cdot r_4}, \ldots$$

where all the r_i satisfy $r_i \geq 2$.

Series - I

Given numbers $x_1, x_2, \ldots, x_n$, the sum $x_1 + x_2 + \cdots + x_n$ is well-defined due to the associative law of addition. Even though some of you have learned to *write* infinite sums as $x_1 + x_2 + \cdots$, it is not immediately clear how this is *defined*! For example, we may be tempted to write the above expression for α as

$$\alpha = \frac{1}{r_2} + \frac{1}{r_2 \cdot r_3} + \frac{1}{r_2 \cdot r_3 \cdot r_4} + \cdots$$

Using the $\sum$ for summations and $\prod$ for products, this can be written even more concisely as

$$\alpha = \sum_{n=2}^{\infty} \frac{1}{\prod_{i=2}^{n} r_i}$$

However, there is a hidden ambiguity in the ∞ symbol that appears above the $\sum$ symbol.

Taking a cue from the above example, let us first limit ourselves to the case when x_i are all positive (or at least non-negative). In that case, the series

$$x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots$$

is an increasing sequence. *If* it is bounded, then we can *define* the infinite sum to be the least upper bound of this sequence. In that case, we say that the *series* $\sum_{i=1}^{\infty} x_i$ is *convergent*. On the other hand, if this sequence is *not* bounded, we say that this series is *divergent*, or more precisely, that it *diverges to infinity*. The term *series* is used for an infinite sum to distinguish from finite sums and from infinite sequences.

It is worth underlining that if a series diverges to infinity, it means that given any number M, there is a natural number $n(M)$ so that $x_1 + x_2 + \cdots + x_n > M$ for *all* $n \geq n(M)$. As we shall see, this *need not* mean that x_n are themselves large.

Geometric Series - I

In the process of making the argument in the previous section, we used the identity

$$(1 - x) \cdot (x + x^2 + \cdots + x^n) = x - x^{n+1}$$

A similar argument (or just adding $(1-x)$ to both sides!) gives the more common identity

$$(1 - x) \cdot (1 + x + \cdots + x^n) = 1 - x^{n+1}$$

When $x \neq 1$ we can divide both sides by $(1 - x)$ to get

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

Now, if $0 < x < 1$, then $0 < 1 - x < 1$. It follows that

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x} < \frac{1}{1 - x}$$

Hence, the sequence

$$1, 1 + x, 1 + x + x^2, \ldots$$

has $1/(1 - x)$ as an upper bound. Moreover, since $x > 0$, it is an increasing sequence, so it has a least upper bound. We shall shortly show that this least upper bound *is* $1/(1 - x)$. For the time being it is enough to note:

> For a number x such that $0 < x < 1$, the sequence with general term $1 + x + \cdots + x^n$ is bounded by $1/(1 - x)$ for all n.

Harmonic Series - I

We now examine the series $\sum_{n=1}^{\infty} 1/n$. Equivalently, we are looking at the sequence of fractions

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots$$

with general term $1 + (1/2) + \cdots + (1/n)$.

We note that $1/3 > 1/4$, so

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Similarly, $1/5$, $1/6$ and $1/7$ are greater than $1/8$, so

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Proceeding similarly, (using induction as appropriate!)

$$\begin{aligned}\sum_{i=1}^{2^k} \frac{1}{2^k+i} &= \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \\ &> \overbrace{\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}^{2^k \text{ times}} = 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}\end{aligned}$$

It follows that

$$\begin{aligned}1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{k+1}} \\ &= 1 + \frac{1}{2} + \sum_{i=1}^{2} \frac{1}{2+i} + \cdots + \sum_{i=1}^{2^k} \frac{1}{2^k+i} \\ &> 1 + \overbrace{\frac{1}{2} + \cdots + \frac{1}{2}}^{k+1 \text{ times}} = 1 + \frac{k+1}{2}\end{aligned}$$

Thus, this series $\sum_{n=1}^{\infty}(1/n)$, whose terms are steadily decreasing, nevertheless diverges to infinity!

Square Harmonic series

Since the series with general term n^2 dominates the series with general term n, we see that $1/n^2$ goes to 0 more rapidly than $1/n$. So, one may imagine that the series $\sum_{i=1}^{\infty} 1/n^2$ could be convergent. As we shall see now, this is indeed the case.

We note that $1/3^2 < 1/2^2$ so

$$\frac{1}{2^2} + \frac{1}{3^2} < \frac{1}{2^2} + \frac{1}{2^2} = 2\frac{1}{2^2} = \frac{1}{2}$$

Similarly,

$$\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} = 4\frac{1}{4^2} = \frac{1}{4}$$

More generally,

$$\sum_{i=0}^{2^k-1} \frac{1}{(2^k+i)^2} = \frac{1}{2^{2k}} + \frac{1}{(2^k+1)^2} + \cdots + \frac{1}{(2^{k+1}-1)^2}$$

$$< \overbrace{\frac{1}{2^{2k}} + \frac{1}{2^{2k}} + \cdots + \frac{1}{2^{2k}}}^{2^k \text{ times}} = 2^k \cdot \frac{1}{2^{2k}} = \frac{1}{2^k}$$

It follows that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(2^{k+1}-1)^2}$$

$$= 1 + \sum_{i=0}^{1} \frac{1}{(2+i)^2} + \cdots + \sum_{i=0}^{2^k-1} \frac{1}{(2^k+i)^2}$$

$$< 1 + \frac{1}{2} + \cdots + \frac{1}{2^k}$$

We have seen above that the last sum is bounded by $1/(1-(1/2)) = 2$. It follows that the sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

is bounded by 2 for *all* n. Hence, the series $\sum_{n=1}^{\infty}(1/n^2)$ converges to a number not bigger than 2. What is this number? It takes some effort to find out!

Zeta function

Since $1/n^k < 1/n^2$ for $k \geq 2$, we can use the above argument to conclude that the series

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

converges for *all* $k \geq 2$. Euler studied these series and found nice formulas for the values when k is even. However, not much is known about the values when k is odd!

More generally, given any sequence $(a_n)_{n\geq 1}$ we can study it by studying the Dirichlet series

$$L((a_n), k) = \sum_{n=1}^{\infty} \frac{a_n}{n^k}$$

If the (a_n) is dominated by polynomial of order p in n, then this converges for $k \geq p+2$.

Some Sequences

Square roots

Given a fraction p/q such that $p^2/q^2 < 5$, we note that $r/s = 5q/p$ is a fraction such that $r^2/s^2 > 5$. So we can form a "combination" $(3p+5q)/(p+3q)$ which lies between p/q and r/s. We note that

$$\begin{aligned}
&(3p+5q)^2 - 5(p+3q)^2 \\
&\qquad = (9p^2+30pq+25q^2) - (5p^2+30pq+45q^2) \\
&\qquad = 4p^2 - 20q^2 = 4(p^2-5q^2) < 0
\end{aligned}$$

Thus $(3p+5q)^2/(p+3q)^2 < 5$. Applying this "process" to $2/1$ we get the sequence

$$\frac{2}{1}, \frac{11}{5}, \frac{29}{13}, \ldots$$

This is a bounded sequence of fractions a/b such that $a^2/b^2 < 5$. It follows easily that $a/b < 3$ for all these fractions. In fact, we will show that the least upper bound of a^2/b^2 *is* 5. Consequently, we will see that the least upper bound of the above sequence is a (positive) number x such that $x^2 = 5$.

More generally, if N is any natural number which is not a square, let k and l be natural numbers such that $k^2 < N$ and $N < l^2$. We take a sequence of rational numbers x_n which starts with $x_1 = k$ and is defined iteratively by the formula

$$x_{n+1} = \frac{lx_n + N}{x_n + l}$$

We note that $x_1^2 = k^2 < N < l^2$. Given that $x_n^2 < N < l^2$, we calculate

$$\begin{aligned}
&(lx_n+N)^2 - N(x_n+l)^2 \\
&\qquad = (l^2x_n^2 + 2lNx_n + N^2) - N\cdot(x_n^2 + 2lx_n + l^2) \\
&\qquad = \left(\left(l^2-N\right)x_n^2 - N\cdot\left(l^2-N\right)\right) \\
&\qquad = (l^2-N)\cdot(x_n^2-N) < 0
\end{aligned}$$

It follows that $x_{n+1}^2 < N < l^2$. This shows by induction that x_{n+1} is bounded and x_{n+1}^2 is bounded by N. It follows that $x_n(x+n+l) = x_n^2 + lx_n < lx_n + N$. Hence $x_n < x_{n+1}$. So,

$$x_1, x_2, x_3, \ldots$$

is a bounded increasing sequence. Thus, it has a least upper bound x. We will show that $x^2 = N$. This method allows us to "construct" square roots of all fractions.

Compounded Interest

On a principal loan amount of P, if a rate of interest r is charged, and interest is "compounded" n times a year, then the total amount to be paid back at the end is

$$P\left(1+\frac{r}{n}\right)^n$$

This is because, you need to pay interest on the accumulated interest as well! If another bank charges interest m times a year, and $m > n$, you may imagine (and we will prove!) that

$$P\left(1+\frac{r}{m}\right)^m > P\left(1+\frac{r}{n}\right)^n \text{ if } m > n$$

As m increases, you are paying more and more! Is there an upper bound to the amount you would have to pay? In other words, we are asking whether the sequence

$$1+r, \left(1+\frac{r}{2}\right)^2, \left(1+\frac{r}{3}\right)^3, \ldots$$

has a least upper bound. We will assume that $r > 0$ since it would be a funny bank which would charge a negative rate of interest!

The Binomial theorem tells us that

$$\left(1+\frac{r}{n}\right)^n = 1 + \binom{n}{1}\frac{r}{n} + \binom{n}{2}\frac{r^2}{n^2} + \cdots \binom{n}{n}\frac{r^n}{n^n}$$

We now use the expression (where $k! = 1 \cdot 2 \cdots k$)

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

to estimate the general term in the right-hand side. First of all, we note

$$\binom{n}{k}\frac{r^k}{n^k} = \frac{r^k \cdot n(n-1)\cdots(n-k+1)}{n^k \cdot k!} = \frac{r^k}{k!}\cdot\left(1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right)$$

If $m > n$, then for any natural number a, we have $a/n > a/m$ and so $(1-a/n) < (1-a/m)$. So we see that

$$\binom{n}{k}\frac{r^k}{n^k} = \frac{r^k}{k!}\cdot\left(1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right)$$
$$< \frac{r^k}{k!}\cdot\left(1\cdot\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{k-1}{m}\right)\right) = \binom{m}{k}\frac{r^k}{m^k}$$

This shows that the sequence above is increasing. Next, we see that

$$\binom{n}{k}\frac{r^k}{n^k} = \frac{r^k}{k!}\cdot\left(1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right) \le \frac{r^k}{k!}$$

It follows that

$$\left(1+\frac{r}{n}\right)^n \leq \sum_{k=0}^{n} \frac{r^k}{k!}$$

At first sight this does not seem to help since we have replaced one complicated expression with another! However, have some faith!

By the Archimedean principle, we know that there is a natural number n_0 so that $n_0 > 2r$. It follows that for $k \geq n_0$

$$\frac{r^k}{k!} = \frac{r^{n_0}}{(n_0)!} \cdot \left(\frac{r}{n_0+1} \cdot \frac{r}{n_0+2} \cdots \frac{r}{k}\right) \leq \frac{r^{n_0}}{(n_0)!} \cdot \frac{1}{2^{k-n_0}}$$

Adding all these up, we see that for $n \geq n_0$

$$\left(1+\frac{r}{n}\right)^n \leq \sum_{k=0}^{n_0-1} \frac{r^k}{k!} + \frac{r^{n_0}}{(n_0)!} \sum_{k=0}^{n-n_0} \frac{1}{2^{k-n_0}}$$

As seen in the study of the geometric series, the last term on the right-hand side is bounded.

$$\sum_{k=0}^{n_0-1} \frac{r^k}{k!} + \frac{r^{n_0}}{(n_0)!} \sum_{k=0}^{n-n_0} \frac{1}{2^{k-n_0}} < \sum_{k=0}^{n_0-1} \frac{r^k}{k!} + \frac{r^{n_0}}{(n_0)!} \cdot 2$$

Since r and n_0 are fixed. This gives an upper bound for our sequence.

In conclusion, (with a lot of work!) we have shown that $(1+r/n)^n$ is a bounded increasing sequence. We will see later that it has a very interesting limit.

At least, as a result of all this work, we can put an upper bound on the amount of interest that the bank can charge us. This should be a relief too!