Growth

One of the reasons we study numbers to understand the answer to "Which is bigger/better?". One typical way is to compare measurements directly. However, many quantities *change* over time. For example, the GDP of a country or the CGPA of a student!

Measurement of a *changing* quantity over time gives rise to a *sequence* since we can think of x_n as representing the measurement after n minutes have elapsed. The study of such "time-series" is fundamental to many sciences. We are interested in how such a sequence grows or declines.

The notion of sequences is a bit outside usual arithmetic. We are specifying a number x_n for each positive integer n. So we are somehow providing *infinitely* many numbers! The usual way to do this is to give a "formula" for x_n as a function of n. Mathematically, there is no difference between the notion of a sequence (x_n) and that of a function $f(n)$ defined for positive integer values of the variable n.

Polynomial Growth

The simplest kind of change is no change at all. (Mathematicians have this habit of starting with trivial and silly examples!) This is represented by a sequence of the form

$$a, a, a, \ldots$$

for some fixed number a.

The next simplest kind is one which changes by a fixed amount b each time. So such a sequence is of the form

$$c, c + b, c + b + b, \ldots$$

Note that the *distributive law* allows us to write the identity

$$2 \cdot b = (1 + 1) \cdot b = 1 \cdot b + 1 \cdot b = b + b$$

Similarly (with a hidden use of "induction* as explained below), we see that adding n copies of b together gives $n \cdot b$ for any natural number n. Hence, the above sequence can also be written as

$$c, c + b, c + 2b, \ldots$$

The first question we can ask is whether this sequence will"overtake" the constant sequence. Intuition says the answer is yes, but how do we *justify* this intuition? First of all, we need the condition that b is a positive number; otherwise these second sequence is *decreasing* rather than increasing! Secondly, our justification *must* take into account the possibility that c and b are *very small* compared with a.

The key idea is the Archimedean principle which tells us that there *is* a natural number n for which $n \cdot b > a - c$. It follows easily that $m \cdot b > a - c$ for *all* $m \geq n$ (another hidden use of induction!). Thus, all the terms of the second sequence from the n-th term onwards are bigger than the first sequence. Note that this does not say much about what n is! To find n we have tue "measure" $(a - c)/b$, or equivalently, measure $(a - c)$ in the unit of length b.

Can we think of something that grows faster than the second sequence? Clearly, increasing b will do the trick. If we take a sequence of the form

$$d, d + e, d + 2e, \ldots$$

where $e > b$, then we have a natural number n so that $n \cdot (e - b) > c - d$. As above, we see that $d + me > c + mb$ for all $m \geq n$.

However, we want something that grows faster than the second sequence *regardless* of how large b is! The formula $c + n \cdot b$ for the second sequence suggests that we (at least) try a formula like $f + n \cdot g + n^2 \cdot h$. As before, for the n^2 term to make a difference, we require h to be positive. Now, take n so that $n \cdot h > (b - g)$. It follows that

$$g + n \cdot h > b$$

By what we have seen above, this means that the sequence (here n is fixed)

$$f, f + (g + n \cdot h), f + 2(g + n \cdot h), \ldots$$

eventually overtakes the sequence

$$c, c + b, c + 2b, \ldots$$

In other words, there is an n_1 for which $f + m(g + n \cdot h) > c + mb$ for all $m \geq n_1$.

On the other hand, if $m \geq n$ we have $g + m \cdot h \geq g + n \cdot n$, so that

$$f + m \cdot g + m^2 \cdot h = f + m(g + m \cdot h) \geq f + m(g + n \cdot h)$$

So, if $m \geq \max\{n, n_1\}$, then we get

$$f + m \cdot g + m^2 \cdot h > f + m(g + n \cdot h) > c + mb$$

A quick summary is to say that quadratic growth *dominates* linear growth. A "physicists" way of saying this is to say that "accelerated" (constant acceleration) growth overtakes steady (constant velocity) growth.

One can imagine that this approach can be extended further. A sequence of the form $a_0 + n \cdot a_1 + \cdots + n^k a_k$ for some fixed integer k and some numbers a_0, a_1, $\ldots$, a_k is called a *polynomial* sequence with *coefficients* a_i. If $a_k > 0$ than we say that this represents *polynomial growth* of degree k. By methods similar to those above (and another use of induction!) we can show that:

> If $k > l$ then polynomial growth of order k *dominates* polynomial growth of order l, *regardless* of the size of the coefficients.

Next, we can ask for a growth pattern that dominates *all* forms of polynomial growth! Before we do that however, we need to sort out "induction" and a few other concepts.

Induction

How can one check an identity or formula for *infinitely* many numbers? After all, we have only a finite amount of time! One important principle that allows us to perform such checks is the "Principle of Induction":

> Given that a certain formula $\phi(n_0)$ is true for a certain natural number n_0 *and* that we can show that if the formula $\phi(n)$ holds, then the formula $\phi(n+1)$ also holds. The *Principle of Induction* allows us to conclude that the formula $\phi(n)$ holds for *all* $n \geq n_0$.

Let us start with a simple application. If $n_0 \cdot b > c$ and $b > 0$ then $n \cdot b > c$ for all $n \geq n_0$. We can see that if $n \cdot b > c$, then, since $(n+1) \cdot b = n \cdot b + b > n \cdot b$, we get $(n+1) \cdot b > c$. (We used the distributive law and transitivity.) Thus, we get our formula $n \cdot b > c$ for all $n \geq n_0$ by the principle of induction.

Of course, in this case, we could have proved it differently by noting that $n - n_0 \geq 0$ implies that $(n - n_0) \cdot b \geq 0$. Adding $n_0 \cdot b$ to both sides and using the distributive law, we get $n \cdot b \geq n_0 \cdot b$. Now using transitivity we get $n \cdot b > c$. However, as will see, there are many examples where the use of the principle of induction is more natural and is often unavoidable.

Binomial coefficients

The binomial coefficient $\binom{n}{r}$ is defined as the number of ways of choosing r objects out of n distinct objects. It is clear that this number is 0 unless $r \leq n$. Moreover, it is clear that $\binom{n}{n} = 1$ and $\binom{n}{0} = 1$.

If we have to choose r objects out of $n+1$ objects, we can do this in two distinct ways:

1. Pick $r-1$ objects out of the first n objects and add the last object to this to get r objects.
2. Pick r objects out of the first n objects.

If the chosen r objects contain the last object of the $n+1$, then this is one of the ways in the first method and otherwise it is one of the ways in the second method. This shows that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

These kinds of "recursive" formulas that express the value at some numbers in terms of values at smaller numbers are a common place where the principle of induction becomes useful.

We can now use induction to prove (for any x) that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$$

First of all, for $n = 1$, this formula becomes

$$(1+x)^1 = 1 + x = \binom{1}{0} + \binom{1}{1}x$$

Next, we check that if the above formula is true, then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)\cdot(1+x)^n = (1+x)^n + x\cdot(1+x)^n \\
&= \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n \\
&\quad + \binom{n}{0}x + \binom{n}{1}x^2 + \cdots + \binom{n}{r}x^{r+1} + \cdots + \binom{n}{n}x^{n+1} \\
&= \binom{n}{0} + \left(\binom{n}{0} + \binom{n}{1}\right)x + \cdots \\
&\quad + \left(\binom{n}{r-1} + \binom{n}{r}\right)x^r + \cdots \\
&\quad + \left(\binom{n}{n-1} + \binom{n}{n}\right)x^n + \binom{n}{n}x^{n+1}
\end{aligned}$$

Now, using the above identities, we get

$$(1+x)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1}x + \cdots + \binom{n+1}{r}x^r + \cdots + \binom{n+1}{n+1}x^n$$

By the principle of induction, we conclude the Binomial theorem.

Sums of Sums of . . .

It is clear from the definition that $\binom{n}{1} = n$. Hence, we have the formula

$$1 + 2 + \cdots + n = \binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1}$$

Using $\binom{1}{1} = \binom{2}{2}$, we can "collapse the terms from the left" using the earlier identity to get

$$1 + 2 + \cdots + n = \binom{n+1}{2}$$

Let us prove this by induction. First of all, for $n = 1$ we have

$$1 = \binom{2}{2}$$

Now, assuming the formula above for a given n, we write

$$1 + 2 + \cdots + n + (n+1)$$
$$= \binom{n+1}{2} + (n+1) = \binom{n+1}{2} + \binom{n+1}{1}$$
$$= \binom{n+2}{2}$$

by using the previous formula. This proves the formula by induction.

We can further generalise this to prove, for any fixed k

$$\binom{1+(k-1)}{k} + \binom{2+(k-1)}{k} + \cdots + \binom{n+(k-1)}{k} = \binom{n+k}{k+1}$$

For $n = 1$, this is

$$\binom{1+(k-1)}{k} = \binom{k}{k} = 1 = \binom{1+k}{k+1}$$

Given the above formula for a particular n, we have

$$\binom{1+(k-1)}{k} + \binom{2+(k-1)}{k} + \cdots + \binom{n+(k-1)}{k} + \binom{(n+1)+(k-1)}{k}$$
$$= \binom{n+k}{k+1} + \binom{n+k}{k}$$
$$= \binom{n+k+1}{k+1} = \binom{(n+1)+k}{k+1}$$

by using the fundamental identity. This proves the formula by induction.

To see the real power of induction, let us do something quite complicated! We define $f_1(n) = 1 + 2 + \cdots + n$ and generalise this to

$$f_{k+1}(n) = f_k(1) + f_k(2) + \cdots + f_k(n)$$

Then $f_{k+1}(n)$ is the sum of the k-fold sums. We claim, by induction, that

$$f_k(n) = \binom{n+(k-1)}{k}$$

We have already proved this for $k = 1$. Assuming it for a particular k, the previous proof shows that it holds for $k+1$ as well. This proves the formula by induction!

Formula for Binomial coefficients

Finally, we use induction to prove a formula for the Binomial coefficients. We claim,

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1}$$

We note that the both sides take the value 1 for $n = k$. We also note that both sides take the value n for $k = 1$.

We now check the formula for $n = 1$ and for all values of $k \geq 1$. The left-hand side is

$$\binom{1}{k} = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases}$$

Now the right-hand side is 1 for $n = k = 1$ as seen above. Moreover, for $k > 1$ and $n = 1$, the right-hand side is

$$\frac{1 \cdot (1-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} = 0$$

Since 0 appears in the numerator.

Now, suppose that the above formula holds for a fixed n and for all $k \geq 1$ and try to prove it for $n+1$ and for all $k \geq 1$. We first note that we have already checked the identity for $k = 1$, so we can assume that $k > 1$. We use the fundamental identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

The right-hand side, by our hypothesis that the formula holds for n becomes

$$\frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} \frac{n \cdot (n-1) \cdots (n-k+2)}{(k-1) \cdots 1}$$

Collecting terms we get

$$\frac{n \cdot (n-1) \cdots (n-k+2)}{k \cdot (k-1) \cdots 1} \cdot ((n-k+1) + k)$$

which simplifies to

$$\frac{(n+1) \cdot n \cdot (n-1) \cdots (n-k+2)}{k \cdot (k-1) \cdots 1}$$

as required.

Exponential Growth

We now return to the question of finding a form of growth that is *faster* than all types of polynomial growth. For any positive number x, consider the sequence

$$(1+x), (1+x)^2, (1+x)^3, \ldots$$

We want to compare this with a polynomial grown of order k given by

$$f(n) = a_0 + n \cdot a_1 + \cdots + n^k \cdot a_k$$

By the Binomial theorem, we see that the n-term is a sum of *positive* terms which contains the expression $\binom{n}{k+1}x^{k+1}$ so that

$$(1+x)^k \geq \binom{n}{k+1} x^{k+1}$$

As seen in the previous subsection

$$\binom{n}{k+1} = \frac{n \cdot (n-1) \cdots (n-k)}{(k+1) \cdot k \cdot (k-1) \cdots 1}$$

The coefficient of n^{k+1} in this expression is $\frac{1}{(k+1) \cdot k \cdots 1}$ which is *positive.* It follows that $\binom{n}{k+1}x^{k+1}$ has polynomial growth of degree $k+1$ (since x is positive). Thus, it *dominates* $f(n)$ as seen earlier.

In summary, we see that $(1+x)^n$ dominates every sequence exhibiting polynomial growth.

We note that, if $g(n) = (1+x)^n$, then

$$g(n+1) = (1+x)g(n) = g(n) + x \cdot g(n)$$

So $g(n+1) - g(n) = x \cdot g(n)$. In words, the amount of growth at between n and $n+1$ is proportional to the value at time n.