Solutions to Assignment 1

1. In each pair of numbers below, which is larger? Explain why.

(1 mark) (a) $10/81$, $11/90$

> **Solution:** We have $81 \cdot 11 = 891$ which is *less* than $10 \cdot 90 = 900$. So $10/81 > 11/90$.

(1 mark) (b) $100/811$, $111/900$

> **Solution:** We have $811 \cdot 111 = 90021$ which is *greater* than $100 \cdot 900 = 90000$. So $100/811 < 111/900$.

2. In each pair of numbers below, which is larger? Explain why.

(1 mark) (a) $(10)^5$, $10000 \cdot (11)^2$

> **Solution:** The first number has 6 digits, while the second one has 7 digits (since $(11)^2 = 121$). So the second one is larger.

(1 mark) (b) $(100)^5$, $10000 \cdot (101)^2$

> **Solution:** The first number has 11 digits, while the second one has 9 digits (since $(101)^2 = 10201$). So the first one is larger.

(1 (bonus)) (c) n^5, $10000 \cdot (n+1)^2$ for large n.

3. Give two positive rational numbers p/q and r/s (this means that p, q, r and s are natural, or counting, numbers). Suppose that $p/q < r/s$, which of the following numbers lies in between?

(1 mark) (a) $\frac{(p/q)+(r/s)}{2}$

> **Solution:** Note that
>
> $$(r/s) - \frac{(p/q) + (r/s)}{2} = \frac{(r/s) - (p/q)}{2} > 0$$
>
> This shows that $(r/s) > \frac{(p/q)+(r/s)}{2}$. The other inequality is similar.

(1 mark) (b) $\sqrt{(pr)/(qs)}$

Solution: Note that

$$\frac{(r/s)}{\sqrt{(pr)/(qs)}} = \frac{\sqrt{r/s}}{\sqrt{p/q}} > 1$$

This shows that $(r/s) > \sqrt{(pr)/(qs)}$. The other inequality is similar.

(1 mark) (c) $(p + r)/(q + s)$

Solution: We are given that $p/q < r/s$. This means $ps < qr$ (since all numbers a positive). Now

$$r(q + s) = rq + rs > ps + rs = s(p + r)$$

It follows that $r/s > (p + r)/(q + s)$. The other inequality is similar.

(1 (bonus)) (d) Order the above three numbers.

4. Given that p and q are counting numbers so that $p^2 > 3q^2$ and put $r/s = (2p+3q)/(p+2q)$. Show that:

(1 mark) (a) $r^2 > 3s^2$

Solution: We just calculate

$$r^2 = 4p^2 + 12pq + 9q^2 \text{ and } 3s^2 = 3p^2 + 12pq + 12q^2$$

Thus, the difference $r^2 - 3s^2$ is $p^2 - 3q^2$ which is given to be positive.

(1 mark) (b) $r/s < p/q$

Solution: We note that $p/q = (2p)/(2q)$. Moreover, given that $p^2 > 3q^2$, we have $p/q > (3q)/p$. So we can use the last part of the previous question to get $p/q > (2p + 3q)/(2q + p)$. It can also be checked by multiplying it out.

(1 (bonus)) (c) Use *this* idea to find a rational number a/b so that $100(a^2 - 3b^2) < b^2$.