Solutions to Assignment 8

1. Solve the classical 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for $x \in [0, \pi]$ under the conditions

$$u(0,t) = u(\pi,t) = 0 \text{ and } u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = 0$$

where f is the function given below. (In this question and subsequent questions you can make use of the Fourier series calculated in other assignments.)

$$f(x) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi \end{cases}$$

Solution: By the method of separation of variables we write the solution in the form $u(x,t) = \sum_n F_n(t)G_n(x)$ where F_n and G_n satisfy the equations

$$\frac{d^2 G_n(x)}{dx^2} + n^2 G_n(x) = 0 \text{ and } G_n(0) = G_n(\pi) = 0$$

and

$$\frac{d^2 F_n(t)}{dt^2} + n^2 a^2 F_n(t) = 0 \text{ and } \frac{dF_n(t)}{dt}(0) = 0$$

These have the solutions

$$G_n(x) = \sin(nx) \text{ and } F_n(t) = a_n \cos(nat)$$

So the solution has the form

$$u(t,x) = \sum_{n=1}^{\infty} a_n \cos(nat) \sin(nx)$$

The given condition says that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

It follows that

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Given the symmetry $x \mapsto \pi - x$ of the function f and the $(-1)^{n+1}$-symmetry of $\sin(nx)$ for this map, we get $a_n = 0$ for $n = 2k$ and for $n = 2k+1$,

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} x \sin((2k+1)x)$$

Using integration by parts we get

$$a_n = \frac{4}{\pi} \left(-x \frac{\cos((2k+1)x)}{2k+1} + \frac{\sin((2k+1)x)}{(2k+1)^2} \right) \Big|_{x=0}^{\pi/2}$$

This gives us

$$a_n = \frac{4 \cdot (-1)^k}{\pi \cdot (2k+1)^2}$$

Thus the complete solution is

$$u(x,t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos((2k+1)at) \sin((2k+1)x)$$

We note that the series is *absolutely convergent* for all x and t. Hence, this "formal" solution is an actual solution.

2. Solve the classical 1-dimensional heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for $x \in [0, \pi]$ and $t \geq 0$ under the conditions

$$u(0,t) = u(\pi,t) = 0 \text{ and } u(x,0) = |\sin(2x)|$$

Solution: By the method of separation of variables we write the solution in the form $u(x,t) = \sum_n F_n(t) G_n(x)$ where F_n and G_n satisfy the equations

$$\frac{d^2 G_n(x)}{dx^2} + n^2 G_n(x) = 0 \text{ and } G_n(0) = G_n(\pi) = 0$$

and

$$\frac{dF_n(t)}{dt} + a^2 n^2 F_n(t) = 0$$

These have the solutions

$$G_n(x) = \sin(nx) \text{ and } F_n(t) = a_n \exp(-a^2 n^2 t)$$

So the solution has the form

$$u(t,x) = \sum_{n=1}^{\infty} a_n \exp(-a^2 n^2 t) \sin(nx)$$

The given condition says that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

It follows that

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Given the symmetry $x \mapsto \pi - x$ of the function f and the $(-1)^{n+1}$-symmetry of $\sin(nx)$ for this map, we get $a_n = 0$ for $n = 2k$ and for $n = 2k+1$,

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} \sin(2x) \sin((2k+1)x)$$

We have the identity

$$\sin(2x) \sin((2k+1)x) = \frac{\cos((2k-1)x) - \cos((2k+3)x)}{2}$$

This gives us

$$a_n = \frac{2}{\pi} \left(\frac{\sin((2k-1)x)}{2k-1} - \frac{\sin((2k+3)x)}{2k+3} \right) \Bigg|_{x=0}^{\pi/2}$$

which simplifies to

$$a_n = \frac{8 \cdot (-1)^{k+1}}{\pi \cdot (2k-1) \cdot (2k+3)}$$

Thus the complete solution is

$$u(x,t) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k-1) \cdot (2k+3)} \exp(-(2k+1)^2 a^2 t) \sin((2k+1)x)$$

We note that the series is *absolutely convergent* for all x and $t \geq 0$. Hence, this "formal" solution is an actual solution.

3. Solve the classical 2-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x^2 + y^2 \leq 1$ under the conditions

$$u(x, y) = x^2 \text{ for } x^2 + y^2 = 1$$

Solution: We convert to polar co-ordinates (r, θ). The problem is then restated as:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

with the boundary condition

$$u(1, \theta) = \cos^2 \theta$$

By the method of separation of variables we write the solution in the form $u(r, \theta) = \sum_n F_n(r) G_n(\theta)$ where F_n and G_n satisfy the equations

$$\frac{d^2 G_n(\theta)}{dx^2} + n^2 G_n(\theta) = 0$$

and

$$\frac{d^2 F_n}{dr^2} + \frac{1}{r}\frac{dF_n}{dr} + \frac{n^2}{r^2} F_n = 0$$

This gives $G_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$ and

$$F_n(r) = \begin{cases} A + B \log(r) & n = 0 \\ Ar^n + Br^{-n} & n > 0 \end{cases}$$

Since $u(r, \theta)$ is finite for $r = 0$ it follows that we can take $F_n(r) = r^n$ for all n. In other words, the solution is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

for constants a_n, b_n such that

$$u(1, \theta) = \cos^2 \theta = \frac{\cos(2\theta) - 1}{2}$$

We see easily that this means $a_n = 0$ for $n \neq 0, 2$ and $a_2 = -a_0 = 1/2$; also, $b_n = 0$ for all n. In other words, the solution is

$$u(x, y) = \frac{x^2 - y^2 + 1}{2} \text{ for } x^2 + y^2 \leq 1$$