Solutions to Assignment 5

1. Use the Power Series method to solve the following differential equations. (Taken from Chapter 3 of Simmons' book on differential equations.)

   (a) $dy/dx = 2xy$. Also solve this as a linear homogeneous equation and compare the solution.

   Solution: If $y = \sum_n a_n x^n$ is the potential power series solution then we obtain, by equating coefficients of x^n, the identity $na_n = 2a_{n-2}$ for $n \geq 2$. We also obtain $a_1 = 0$. There is no condition on a_0. We obtain $a_2 = a_0$, $a_3 = 2a_1/3 = 0$, and more generally

   $$a_n = \begin{cases} \frac{1}{k!}a_0 & n = 2k \\ 0 & n = 2k+1 \end{cases}$$

   The solution of the linear homogeneous equation is $c\exp(x^2)$ which agrees with this by writing out the power series for $\exp(x^2)$ and putting $c = a_0$

   (b) $dy/dx = x - y$ and $y(0) = 0$. Also solve this as a linear inhomogeneous equation and compare the solution.

   Solution: If $y = \sum_n a_n x^n$ is the potential power series solution then we obtain, by equating coefficients of x^n, the identity $na_n = -a_{n-1}$ for $n \geq 3$. We also obtain $a_1 = -a_0$ and $2a_2 = 1 - a_1$. There is no condition on a_0. Putting $a_0 = y(0) = 0$, we obtain $a_2 = 1$, $a_3 = -(1)/3$, and more generally

   $$a_n = \begin{cases} \frac{(-1)^k}{k!} & n \geq 2 \\ 0 & n = 1 \end{cases}$$

   The solution of the linear inhomogeneous equation is $(x-1) + \exp(-x)$ which agrees with this by writing out the power series for $\exp(-x)$.

   (c) $dy/dx = 1+y^2$. It may not be easy to find a recursion formula for the n-th coefficient so just calculate the first 5 coefficients.

   Solution: Assuming $y = \sum_n a_n x^n$, we see that $a_n = y^{(n)}(0)/(n!)$; where we use the notation $y^{(n)} = d^n y/dx^n$.
   We repeatedly differentiate the differential equation to obtain

   $$\begin{aligned} y^{(1)} &= 1 + y^2 \\ y^{(2)}(0) &= 2yy^{(1)} & &= 2y + 2y^3 \\ y^{(3)}(0) &= (2+6y^2)y^{(1)} & &= 2 + 8y^2 + 6y^4 \\ y^{(4)}(0) &= (16y + 24y^3)y^{(1)} & &= 16y + 40y^3 + 24y^5 \end{aligned}$$

Using $y(0) = c$, we obtain

$$y = c + (1+c^2)x + c(1+c^2)x^2 + (1+4c^2+6c^4)(x^3/3) + (2c+5c^3+3c^5)(x^4/3) + \cdots$$

(d) $d^2y/dx^2 = -xy$.

Solution: If $y = \sum_n a_n x^n$ is the potential power series solution then we obtain, by equating coefficients of x^n, the identity $n(n-1)a_n = -a_{n-3}$ for $n \geq 3$ and $a_2 = 0$. There is no condition on a_0 or a_1. We obtain for all $k \geq 0$,

$$a_n = \begin{cases} \frac{(-1)^k}{(3k)(3k-1)(3k-3)(3k-4)\cdots} a_0 & n = 3k \\ \frac{(-1)^k}{(3k+1)(3k)(3k-2)(3k-3)\cdots} a_1 & n = 3k+1 \\ 0 & n = 3k+2 \end{cases}$$

(e) $(d^2y/dx^2) - 2x(dy/dx) + 2ay = 0$.

Solution: If $y = \sum_n a_n x^n$ is the potential power series solution then we obtain, by equating coefficients of x^n, the identity $n(n-1)a_n - 2(n-2)a_{n-2} + 2aa_{n-2} = 0$ for $n \geq 2$. There is no condition on a_0 or a_1. We obtain for all $k \geq 0$,

$$a_n = \begin{cases} 2^k \frac{a(a-2)\cdots(a-2k+2)}{(2k)!} a_0 & n = 2k \\ 2^k \frac{(a-1)(a-3)\cdots(a-2k+1)}{(2k+1)!} a_1 & n = 2k+1 \end{cases}$$

In particular, we note that if a is an integer then one of the solutions is a polynomial.