Solutions to Assignment 4

1. Determine with of the following are exact and solve those which are exact. (Taken from Section 2.8 of Simmons' book on differential equations.)

   (a) $ydx + (x + 2/y)dy = 0$.

   Solution: We have $(M, N) = (y, x + 2/y)$. We check

   $$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

   So we can put

   $$\Phi(x, y) = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx \right) dy$$

   which we calculate as

   $$\Phi(x, y) = xy + 2 \log y$$

   (b) $(y - x^2)dx + (x + y^2)dy = 0$.

   Solution: We have $(M, N) = (y - x^2, x + y^2)$. We check

   $$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

   So we can put

   $$\Phi(x, y) = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx \right) dy$$

   which we calculate as

   $$\Phi(x, y) = xy - x^3/3 + y^3/3$$

   (c) $(\cos x \cos^2 y)dx + 2(\sin x \sin y \cos y)dy = 0$.

   Solution: We have $(M, N) = (\cos x \cos^2 y, 2(\sin x \sin y \cos y)$. We check

   $$\frac{\partial M}{\partial y} = -2 \cos x \sin y \cos y \neq 2 \cos x \sin y \cos y = \frac{\partial N}{\partial x}$$

   So this is not exact. On the other hand we see that the following equation *is* exact.

   $$(\cos x \cos^2 y)dx - 2(\sin x \sin y \cos y)dy = 0$$

For this equation we calculate easily (or by inspection!)

$$\Phi(x, y) = \sin x \cos^2 y$$

(d) $(1 + y)dx + (1 - x)dy = 0$.

Solution: We have $(M, N) = (1 + y, 1 - x)$. We check

$$\frac{\partial M}{\partial y} = 1 \neq -1 = \frac{\partial N}{\partial x}$$

So this is not exact. On the other hand we see that the following equation *is* exact.

$$(1 + y)dx - (1 - x)dy = 0$$

We get easily by inspection that

$$\Phi(x, y) = x - y + xy$$

2. Solve each of the following equations by finding an integration factor. (Taken from Section 2.9 of Simmons' book on differential equations.)

(a) $(3x^2 - y^2)dx - (2xy)dy = 0$.

Solution: We have $(M, N) = (3x^2 - y^2, -2xy)$. We check

$$\frac{\partial M}{\partial y} = -2y = \frac{\partial N}{\partial x}$$

So we can take the integration factor as 1!

$$\Phi(x, y) = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx \right) dy$$

which we calculate as

$$\Phi(x, y) = x^3 - xy^2$$

(b) $ydx + (x + 3x^3y^4)dy = 0$.

Solution: We have $(M, N) = (y, x + 3x^3y^4)$. We check

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - (1 + 9x^2y^4) = -9x^2y^4$$

To find the integration factor q, we need to solve the equation

$$\frac{\partial q}{\partial x}(x + 3x^3y^4) - \frac{\partial q}{\partial y}y = (-9x^2y^4)q$$

This looks difficult. However, we note that if $q = x^a y^a$, then

$$\frac{\partial q}{\partial x} = ax^{a-1}y^a$$
$$\frac{\partial q}{\partial y} = ax^a y^{a-1}$$

Substituting, we obtain

$$ax^a y^a(1 + 3x^2y^4) - ax^a y^a = (-9x^2y^4)(x^a y^a)$$

which simplifies to $3ax^{a+2}y^{a+4} = -9x^{a+2}y^{a+4}$. This has the solution $a = -3$. So, we put the integration factor as $q = 1/(xy)^3$.

$$\Phi(x, y) = \int qM dx + \int \left(qN - \frac{\partial}{\partial y}\int qM dx\right) dy$$

which we calculate as

$$\Phi(x, y) = \frac{-1}{2x^2y^2} + \frac{1}{2y^2}$$

(c) $(x - y)dx + (x + y)dy = 0$.

Solution: We have $(M, N) = (x - y, x + y)$. We check

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2$$

To find the integration factor q, we need to solve the equation

$$\frac{\partial q}{\partial x}(x + y) - \frac{\partial q}{\partial y}(x - y) = -2q$$

This looks difficult. On the other hand we *can* solve the linear equations with constant coefficients

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = y - x$$

These are the parametric equations of the curves $\Phi(x, y) = c$ that we are looking for. The parametric curve is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t\,(a\cos t + b\sin t) \\ e^t\,(-a\sin t + b\cos t) \end{pmatrix}$$

It follows that $\frac{x(t)}{y(t)} = \tan(t + \theta)$ where $\tan\theta = a/b$. In other words, we see that $t = \tan^{-1}(x/y) - \theta$. Further, we have $x^2 + y^2 = e^{2t}(a^2 + b^2)$, so that $t = (1/2)\log(x^2 + y^2) - \log r$ for $r^2 = (a^2 + b^2)$. It follows that

$$\Phi(x, y) = (1/2)\log(x^2 + y^2) - \tan^{-1}(x/y)$$

is the function whose level curves are the trajectories. (A more geometric approach is to note that these are the logarithmic spirals.)