Solutions to the Questions

(4 marks) 1. Indicate which of the following is exact and solve the one which is exact.

(a) $(e^x + y)dx + (x + \cos y)dy = 0$.

Solution: We have $(M, N) = (e^x + y, x + \cos y)$. We check

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

So this is exact. (1 Mark) So we can put

$$\Phi(x, y) = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx \right) dy$$

(1 Mark for this formula or with role of (M, x) and (N, y) reversed). which we calculate as

$$\Phi(x, y) = e^x + xy + \sin y$$

(1 Mark for this answer.)

(b) $(\sin x \sin^2 y)dx + 2(\cos x \cos y \sin y)dy = 0$.

Solution: We have $(M, N) = (\sin x \sin^2 y, 2(\cos x \cos y \sin y)$. We check

$$\frac{\partial M}{\partial y} = 2 \sin x \sin y \cos y \neq -2 \cos x \cos y \sin y = \frac{\partial N}{\partial x}$$

So this is not exact. (1 Mark)

2. For each of the following equations, find an integration factor as indicated.

(2 marks) (a) Integration factor of $ydx + 2xdy = 0$ as a function of y.

Solution: We have $(M, N) = (y, 2x)$. We check

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - 2 = -1$$

If we assume that the factor of integration is a function q of y alone, then we have to solve for

$$\frac{dq}{dy} M + q \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = 0$$

This is the equation (1 Mark for this.)

$$y \frac{dq}{dy} - q = 0$$

This has the solution $q = y$. (1 Mark for this.)

(2 marks) (b) Integration factor of $ydx + 2xdy = 0$ as a function of x.

Solution: Using q as a function of x, we obtain the equation

$$-\frac{dq}{dx}N + q\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0$$

This is the equation (1 Mark for this.)

$$-2x\frac{dq}{dx} - q = 0$$

which has the solution $q = 1/\sqrt{x}$. (1 Mark for this.)

(1 mark) (c) Integration factor of $ydx + 2xdy = 0$ which is a function of x and y. (Dependent on both.)

Solution: Since $y(ydx+2xdy) = dF$ for some F and $(1/\sqrt{x})(ydx+2xdy) = dG$ for some G it follows that

$$\left(y + \frac{1}{\sqrt{x}}\right)(ydx + 2xdy) = d(F + G)$$

So $q(x, y) = y + 1/\sqrt{x}$ is also an integration factor.
A different approach is to note that the equation has the form $f(y)dx+g(x)dy = 0$. So dividing by $f(y)g(x)$ will give $dx/g(x)+dy/f(y)$ which is obviously exact. So we can take $q(x, y) = 1/xy$.

3. Use the Power Series method to solve the following differential equations upto the term involving x^3 in the form

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

by finding a_0, a_1, a_2 and a_3.

(3 marks) (a) $(dy/dx) = x - y$ with $y(0) = 1$.

Solution: We inductively calculate to obtain the equations

$$y^{(1)} = x - y$$
$$y^{(2)} = 1 - x + y$$
$$y^{(3)} = -1 + x - y$$

Substituting $y(0) = 1$ and using Taylor series we obtain

$$y(x) = 1 - x + x^2 - x^3/3 + \dots$$

(1 mark for each a_i except a_0.)

(3 marks) (b) $(dy/dx) = e^y$ with $y(0) = 1$.

Solution: We inductively calculate to obtain the equations

$$y^{(1)} = e^y$$
$$y^{(2)} = e^{2y}$$
$$y^{(3)} = e^{3y}$$

Substituting $y(0) = 1$ and using Taylor series we obtain

$$y(x) = 1 + ex + e^2x^2/2 + +e^3x^3/6 + \dots$$

(1 mark for each a_i except a_0.)