Algebraic tricks

Some algebraic tricks can be used to find integration factors in some special cases. The derivations given below are *best* understood by solving some *explicit* example while following these notes.

If we have $d\Phi = Mdx + Ndy$, then

$$d(\Phi^a) = a\Phi^{a-1}d\Phi = a\Phi^{a-1}\,(Mdx + Ndy)$$

Secondly, If $\Psi(x,y) = f(x) + g(y)$ then

$$d\Psi = \frac{df}{dx}dx + \frac{dg}{dy}dy$$

Combining these two identities we see that

$$d((1/a)\Phi^a + \Psi) = \Phi^{a-1}\left(\left(M + \frac{\frac{df}{dx}}{\Phi^{a-1}}\right)dx\left(N + \frac{\frac{dg}{dy}}{\Phi^{a-1}}\right)dy\right)$$

In other words, $q = \Phi^{a-1}$ is an integration factor for the differential

$$\left(M + \frac{\frac{df}{dx}}{\Phi^{a-1}}\right)dx\left(N + \frac{\frac{dg}{dy}}{\Phi^{a-1}}\right)dy$$

This idea will be applied in two special cases below.

Equations like $ydx + xdy = px^ry^sdx$

In this case, we take $\Phi(x,y) = xy$ so that $d\Phi = ydx + xdy$. We now need to choose a suitable f so that

$$\frac{\frac{df}{dx}}{xy^{a-1}} = -px^ry^s$$

Equivalently, we have

$$\frac{df}{dx} = -px^{r+a-1}y^{s+a-1}$$

It follows that we need $a = -s+1$, and we can take $f(x) = \frac{-p}{r+a}x^{r+a}$. In other words, we take the factor of integration as $(xy)^{-s}$. The equation then becomes

$$x^{-s}y^{-s+1}dx + x^{-s+1}y^{-s}dy = px^{r-s}dx$$

It is simple (assuming $s \neq 1$ and $r - s \neq -1$) to see this as

$$\frac{1}{s-1}d(x^{1-s}y^{1-s}) = \frac{p}{r+1-s}d\left(x^{r+1-s}\right)$$

In other words, we have

$$\Phi(x,y) = \frac{1}{s-1}(xy)^{1-s} - \frac{p}{r+1-s}x^{r+1-s}$$

A similar argument can be used for the cases when $s = 1$ or $r - s = -1$. Also note that we can interchange x and y to solve $ydx + xdy = px^r y^s dy$ in a similar way.

Equations like $ydx - xdy = px^r y^s dx$

In this case, we take $\Phi(x,y) = x/y$ so that $d\Phi = (ydx - xdy)/y^2$. We now need to choose a suitable f so that

$$\frac{\frac{df}{dx}}{x^{a-1}y^{-1-a}} = -px^r y^s$$

Equivalently, we have

$$\frac{df}{dx} = -px^{r+a-1}y^{s-1-a}$$

It follows that we need $a = s-1$, and we can take $f(x) = \frac{-p}{r+a}x^{r+a}$. In other words, we take the factor of integration as x^{a-1}/y^{a+1}. The equation then becomes

$$\frac{x^{s-2}}{y^{s-1}}dx - \frac{x^{s-1}}{y^s}dy = px^{r+s-2}dx$$

It is simple (assuming that $s \neq 1$ and $r + s \neq -1$) to see this as

$$\frac{1}{s-1}d\left(\frac{x^{s-1}}{y^{s-1}}\right) = \frac{p}{r+s-1}d\left(x^{r+s-1}\right)$$

In other words, we have

$$\Phi(x,y) = \frac{1}{s-1}\frac{x^{s-1}}{y^{s-1}} - \frac{p}{r+s-1}x^{r+s-1}$$

A similar argument can be used for the cases when $s = 1$ or $r+s = 2$. Also note, as before, that we can interchange the roles of x and y to solve $ydx - xdy = px^r y^s dy$ in a similar way.