Differential Equations for Scientists

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Plane curves and flows

A family of plane curves is described by equations of the type $\Phi(x, y) = c$ where Φ is a "nice" function of two variables and c is the parameter that chooses a particular curve in the family. If $(x(t), y(t))$ is a parametric solution to such a curve, then we have $f(t) = \Phi(x(t), y(t)) = c$ is a constant. Hence

$$0 = \frac{df}{dt} = \frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt}$$

If we put

$$(M(x, y), N(x, y)) = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right)$$

then this becomes $M\dot{x}(t) + N\dot{y}(t) = 0$ which says that the tangent vector $(\dot{x}(t), \dot{y}(t))$ to the curve is perpendicular to the vector (M, N). In other words, the tangent vector is a multiple of $(N, -M)$.

Conversely, given a vector $(N, -M)$ at each point of the plane, we can ask whether there is a family of curves which have these vectors as tangent vectors. This problem can be solved if we find a function Φ such that

$$(M(x, y), N(x, y)) = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right)$$

This question is sometimes posed (in notation which we will not explain here!) as solving for Φ in the equation

$$d\Phi = M dx + N dy$$

If we find such a function, then the equation $\Phi(x, y) = c$ can be seen as *implicitly* defining y as a function of x. For example, the equation $x^2 + y^2 = r^2$ can be seen as implicitly defining $y = \sqrt{r^2 - x^2}$. Thus, the problem is also sometimes posed as the problem of solving the differential equation $M + N\frac{dy}{dx} = 0$.

In other words, we see that given a pair of functions (M, N) of two variables, the following three problems are related:

1. Find a function Φ such that $(M, N) = (\partial\Phi/\partial x, \partial\Phi/\partial y)$.

2. Find a family of curves $\Phi(x, y) = c$ whose tangent at (x, y) is parallel (or proportional) to $(N, -M)$.
3. Solve the differential equation $M + N dy/dx = 0$ to write y as an implicit function of x.

These are *not exactly* the same problem. We can see that if we find a function Φ such that

$$\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right) = (qM, qN)$$

for a suitable (non-zero!) function q, then we have a solution of the second problem. Moreover, finding such a Φ also gives an implicit solution for the third problem.

Exact and closed differentials

If we have a solution Φ for the first problem, we see that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial^2}{\partial x \partial y}\Phi - \frac{\partial^2}{\partial y \partial x}\Phi = 0$$

In other words, the first problem *only* has a solution, *if* the condition

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

holds. In such a case we say that the differential $Mdx + Ndy$ is "closed" or "locally exact"[1]. We can then put

$$\Phi(x, y) - \Phi(0, y) = \int_0^x M dx$$

where we integrate with respect to x treating y as a constant. Differentiating both sides with respect to y we have

$$N - \frac{d\Phi(0, y)}{dy} = \frac{\partial}{\partial y}\left(\int_0^x M dx\right)$$

In other words, we get

$$\frac{d\Phi(0, y)}{dy} = N - \frac{\partial}{\partial y}\left(\int_0^x M dx\right)$$

[1] In most (classical) books on this topic you will find that this condition is called "exact". However, in terms of modern conventions, it is better to refer to it as "closed" or "locally exact".

In fact, we can check that the right-hand side *does not depend* on x

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\left(\int_0^x M dx\right)\right)$$
$$= \frac{\partial}{\partial x}N - \frac{\partial^2}{\partial x \partial y}\left(\int_0^x M dx\right)$$
$$= \frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M = 0$$

It follows that

$$\Phi(0, y) - \Phi(0, 0) = \int_0^y \left(N - \frac{\partial}{\partial y}\left(\int_0^x M dx\right)\right) dy$$

Note that we only *want* Φ up to a constant. So, putting it all together, we get a formula

$$\Phi(x, y) = c + \int_0^x M dx + \int_0^y \left(N - \frac{\partial}{\partial y}\left(\int_0^x M dx\right)\right) dy$$

This formula gives us Φ whenever the differential $M dx + N dy$ is "closed" or "locally exact".

In fact, one can check that given *any* path from $(0, 0)$ to (x, y) we can "integrate along the path" to get $\Phi(x, y) - \Phi(0, 0)$. So we can also give a more "symmetric" formula

$$\Phi(x, y) = c + \int_0^1 (M(xt, yt)x + N(xt, yt)y)\, dt$$

where the integration is to be performed with respect to t keeping x and y as constants.

Inexact differentials

We have seen that the differential equation $dy/dx = -M/N$ has a solution with given initial conditions whenever the right-hand side is a "sufficiently nice" function. Now, there may be points where $N = 0$ causing this equation to "blow up", but as long as $M \neq 0$ at and near those points we can solve $dx/dy = -N/M$ to find x as a function of y. Putting these "bits of curves" together, we can find curves at all points where $(M, N) \neq (0, 0)$.

A different approach is to consider the pair of differential equations

$$\frac{dx}{dt} = N(x, y)$$
$$\frac{dy}{dt} = -M(x, y)$$

as a single vector-valued equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} N(x,y) \\ -M(x,y) \end{pmatrix}$$

As seen earlier, for "sufficiently nice" functions M and N, this can be integrated for each initial value (x_0, y_0) to give a flow for a value of t not too large. It follows that we have a "flow" at every point of the plane. Since the flow is time-independent, the flow is along plane curves.

In summary, we can justify the assertion that there is a function $\Phi(x, y)$ such that the solutions of the above differential equations "lie along" the curves $\Phi(x, y) = c$. Now the tangent to this curve at (x, y) is parallel to the vector $(\partial\Phi/\partial y, -\partial\Phi/\partial x)$. Since the slope of the tangent to the curves considered above is $-N/M$, we see that

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= q(x,y)N(x,y) \\ -\frac{\partial \Phi}{\partial x} &= -q(x,y)M(x,y) \end{aligned}$$

for a suitable function q. This is equivalent to the assertion that, for this choice of q the differential $q(Mdx + Ndy)$ is closed. In other words, there is a function Φ such that

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= q(x,y)M(x,y) \\ \frac{\partial \Phi}{\partial y} &= q(x,y)N(x,y) \end{aligned}$$

So, if $Mdx + Ndy$ is *any* differential, then there is a suitable function q so that $q(Mdx + Ndy)$ which is closed or "locally exact". In classical literature, this q is called the "integration factor". Unfortunately, it is *not* unique which makes it difficult to find! Note that once we have found q, we can use the methods of the previous section to find Φ.

Let us first see what equation is satisfied by q. By the closed condition, we have

$$\frac{\partial}{\partial x}(qN) - \frac{\partial}{\partial y}(qM) = 0$$

This gives us

$$N\frac{\partial q}{\partial x} - M\frac{\partial q}{\partial y} = q\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

Since we have not learnt any techniques to solve partial differential equations so far, this seems like a hopeless case! However, it *may* be the case that q depends *only* on x. In that case, we obtain the equation

$$\frac{dq}{dx} = q\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

We *can* solve this to get the formula

$$q = \exp\left(\int dx \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)$$

Obviously, this formula *only* works if the expression

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

depends *only* on x. Since M and N are known to us, we can calculate and check this condition.

A similar formula can be found for q as a function of y, if the following expression depends *only* on y.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

Thus, the above equation for q *can* be solved in some special cases. In general, one can only prove a result to the effect that a solution exists!

An example

We end with an "artificial example" that demonstrates the above techniques.

Consider the differential $dx + (x + 2ye^{-y})dy$. We see that $(M, N) = (1, x + 2ye^{-y})$ and

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -1 \neq 0$$

So we need to find an integration factor. We can solve

$$\frac{dq}{dy} = q \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -q$$

and note that the solution $q = e^y$ depends only on y as required. So we take $(M_1, N_1) = (e^y, x + 2y)$ and solve for Φ as usual.

$$\Phi(x, y) = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int N dx\right) dt = xe^y + \int (xe^y + 2y - xe^y)\, dy = xe^y + y^2$$

You can find many more examples in section 2.8 and 2.9 of Simmons' book on differential equations.