Differential Equations for Scientists

IDC205 First Mid-Sem Exam 11 September 2018

Solutions to the Questions

1. What is $\exp(tA)$ for each of the following matrices.

(2 marks) (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Solution: This is a diagonal matrix, so $\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$

(2 marks) (b) $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: This is a nilpotent matrix ($A^3 = 0$), so it is $\mathbf{1}_2 + tA + t^2A^2/2$. $\begin{pmatrix} 1 & -t & t - t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}$

(2 marks) (c) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Solution: This matrix is of the form $a\mathbf{1}_2+bI$ so it is $e^tR(t) = \begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix}$

2. Solve the following Linear ODE Initial value problem in the steps given below.

$$\frac{dx}{dt} = -2x + y + t$$
$$\frac{dy}{dt} = -4x + 2y$$

with $x(0) = 0$ and $y(0) = 1$.

(2 marks) (a) Write the equation in the form

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v} + \vec{w} \text{ and } \vec{v}(0) = \vec{v_0}$$

where $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and A is a 2×2 matrix. What are A, $\vec{w}$ and $\vec{v_0}$?

Solution:

$$A = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \; ; \; \vec{w} = \begin{pmatrix} t \\ 0 \end{pmatrix} \; ; \; \vec{v_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(2 marks) (b) Find a matrix-valued function $G(t)$ so that

$$\frac{dG}{dt} = A \cdot G \text{ and } G(0) = \mathbf{1}_2$$

Solution: We just need to write $\exp(tA)$ since A has constant coefficients. Moreover $A^2 = 0$ so it is

$$G = \mathbf{1}_2 + tA = \begin{pmatrix} 1-2t & t \\ -4t & 1+2t \end{pmatrix}$$

(2 marks) (c) Find a vector-valued function $\vec{u}(t)$, so that $\vec{v} = G \cdot \vec{u}$ is a solution of the above initial value problem.

Solution: We need to satisfy the equations

$$G \cdot \frac{d\vec{u}}{dt} = \vec{w} \text{ and} \vec{u}(0) = \vec{v_0}$$

So the solution is

$$\vec{u} = \vec{v_0} + \int_0^t G(t)^{-1} \vec{w} dt$$

Using

$$G^{-1} = \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}$$

we see that

$$\vec{u} = \begin{pmatrix} t^2/2 + 2t^3/3 \\ 1 + 4t^3/3 \end{pmatrix}$$

3. Answer each of the following questions and provide a reason.

(1 mark) (a) Are $(\cos(t), \sin(t))$ and $(t, 1-t)$ two solutions of the *same* differential equation

$$\frac{d\vec{v}}{dt} = \vec{f}(\vec{v})$$

for 2-dimensional vector-valued functions $\vec{v}$ and $\vec{f}$ for different initial conditions?

Solution: The first solution passes through $(1, 0)$ at $t = 0$ and the second solution passes through the same point $t = 1$. However, these solutions are different *around* these values of t. This contradicts the unique-ness of the solution. (The question should have mentioned that $\vec{f}$ is continuous and $\vec{v}$ is differentiable!)

(1 mark) (b) Is $(\cos t, t \sin t)$ a possible solution for a differential equation of the form

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v}$$

where $\vec{v}$ is a 2-dimensional vector and

$$A = \begin{pmatrix} 0 & -f(t) \\ f(t) & 0 \end{pmatrix}$$

for some suitable function $f(t)$?

Solution: The vector $(\cos t, t \sin t)$ changes in length with t. This is not possible since A is a skew-symmetric matrix so the flow preserves lengths and angles. (The question should have mentioned that f is continuous.)

(1 mark) (c) Is $G(t) = \begin{pmatrix} t & t^2 + 1 \\ -1 & 1 \end{pmatrix}$ a matrix solution of a differential equation of the form

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v}$$

where $\vec{v}$ is a 2-dimensional vector and

$$A = \begin{pmatrix} h(t) & f(t) \\ g(t) & -h(t) \end{pmatrix}$$

for some suitable functions $f(t)$, $g(t)$ and $h(t)$?

Solution: Since the matrix A has trace 0, the matrix G must have constant determinant. However, the given matrix G has determinant $t + t^2 + 1$ which varies with t. (The question should have mentioned that f , g and h are continuous.)