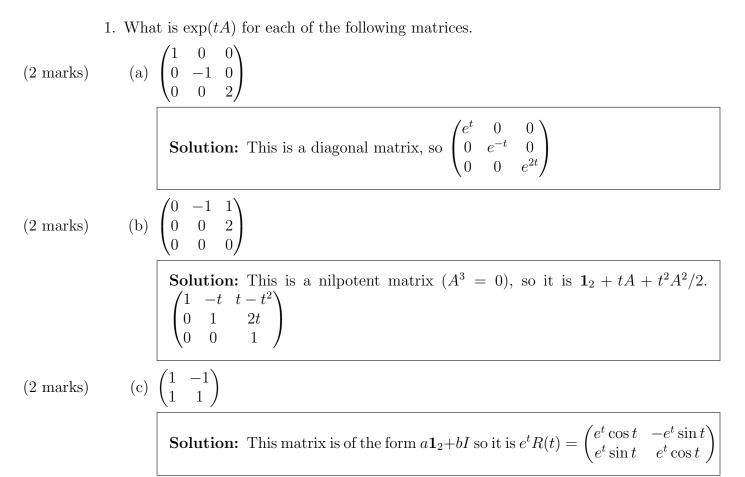
Solutions to the Questions



2. Solve the following Linear ODE Initial value problem in the steps given below.

$$\frac{dx}{dt} = -2x + y + t$$
$$\frac{dy}{dt} = -4x + 2y$$

with x(0) = 0 and y(0) = 1.

(2 marks)(a) Write the equation in the form

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v} + \vec{w} \text{ and } \vec{v}(0) = \vec{v_0}$$

where
$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$
 and A is a 2 × 2 matrix. What are A, \vec{w} and $\vec{v_0}$?

Solution:

$$A = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \; ; \; \vec{w} = \begin{pmatrix} t \\ 0 \end{pmatrix} \; ; \; \vec{v_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(2 marks)

(b) Find a matrix-valued function G(t) so that

$$\frac{dG}{dt} = A \cdot G \text{ and } G(0) = \mathbf{1}_2$$

Solution: We just need to write $\exp(tA)$ since A has constant coefficients. Moreover $A^2 = 0$ so it is

$$G = \mathbf{1}_2 + tA = \begin{pmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{pmatrix}$$

(2 marks)(c) Find a vector-valued function $\vec{u}(t)$, so that $\vec{v} = G \cdot \vec{u}$ is a solution of the above initial value problem.

Solution: We need to satisfy the equations

$$G \cdot \frac{d\vec{u}}{dt} = \vec{w} \text{ and } \vec{u}(0) = \vec{v_0}$$

So the solution is

$$\vec{u} = \vec{v_0} + \int_0^t G(t)^{-1} \vec{w} dt$$

Using

we see that

$$G^{-1} = \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}$$
$$\vec{u} = \begin{pmatrix} t^2/2 + 2t^3/3 \\ 1+4t^3/3 \end{pmatrix}$$

3. Answer each of the following questions and provide a reason.

(1 mark)(a) Are $(\cos(t), \sin(t))$ and (t, 1-t) two solutions of the same differential equation

$$\frac{d\vec{v}}{dt} = \vec{f}(\vec{v})$$

for 2-dimensional vector-valued functions \vec{v} and \vec{f} for different initial conditions?

Solution: The first solution passes through (1,0) at t = 0 and the second solution passes through the same point t = 1. However, these solutions are different *around* these values of t. This contradicts the unique-ness of the solution. (The question should have mentioned that \vec{f} is continuous and \vec{v} is differentiable!)

(1 mark) (b) Is $(\cos t, t \sin t)$ a possible solution for a differential equation of the form

$$\frac{d\vec{v}}{dt} = A\cdot\vec{v}$$

where \vec{v} is a 2-dimensional vector and

$$A = \begin{pmatrix} 0 & -f(t) \\ f(t) & 0 \end{pmatrix}$$

for some suitable function f(t)?

Solution: The vector $(\cos t, t \sin t)$ changes in length with t. This is not possible since A is a skew-symmetric matrix so the flow preserves lengths and angles. (The question should have mentioned that f is continuous.)

(1 mark) (c) Is
$$G(t) = \begin{pmatrix} t & t^2 + 1 \\ -1 & 1 \end{pmatrix}$$
 a matrix solution of a differential equation of the form

$$\frac{d\vec{v}}{dt} = A \cdot \bar{v}$$

where \vec{v} is a 2-dimensional vector and

$$A = \begin{pmatrix} h(t) & f(t) \\ g(t) & -h(t) \end{pmatrix}$$

for some suitable functions f(t), g(t) and h(t)?

Solution: Since the matrix A has trace 0, the matrix G must have constant determinant. However, the given matrix G has determinant $t+t^2+1$ which varies with t. (The question should have mentioned that f, g and h are continuous.)