Solutions to Quiz 3

(5 marks) 1. Find the solution of the differential equation

$$\frac{dx}{dt} = x - ty + 1$$
$$\frac{dy}{dt} = y$$

with initial condition $(x(0), y(0)) = (0, 1)$

Solution: We note that the equation has the matrix form

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \cdot \vec{v} + \vec{w}$$

where $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (1 mark for this step.)

We note that the matrix function $A(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ has the property that $A(t)$ commutes with $A(t')$ for all t and t'. It follows that the general solution of the *homogeneos* equation is given by

$$\vec{v} = \exp\left(\int_0^t A(t)dt\right) \cdot \vec{v_0} = B(t) \cdot \vec{v_0}$$

(1 mark for this step.) We calculate

$$B(t) = \exp\left(\begin{pmatrix} t & -t^2/2 \\ 0 & t \end{pmatrix}\right) = \exp\left(t\mathbf{1}_2 + \begin{pmatrix} 0 & -t^2/2 \\ 0 & 0 \end{pmatrix}\right) = \exp(t)\begin{pmatrix} 1 & -t^2/2 \\ 0 & 1 \end{pmatrix}$$

(1 Mark for this step.)

Using variation of parameters method, we see that the solution to the homogeneous equation is given by

$$\vec{v} = B(t) \cdot \vec{u}$$

where

$$\vec{u} = \vec{v_0} + \int_0^t \left(B(t)^{-1}\vec{w}\right) dt$$

(1 Mark for this step.)

We calculate using the given initial vector $\vec{v_0}$ and $\vec{w}$ as above

$$\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t B(t)^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \exp(-t) \\ 0 \end{pmatrix} dt = \begin{pmatrix} 1 - \exp(-t) \\ 1 \end{pmatrix}$$

(1 mark for this part.)

Thus, the final solution is

$$\vec{v} = \begin{pmatrix} e^t - 1 - (t^2/2)e^t \\ e^t \end{pmatrix}$$