Solutions to Assignment 3

1. Solve the following ordinary differential equation

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v} + \vec{f}(t) \text{ and } \vec{v}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for each of the following choices of A and f.

(a) $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $\vec{f}(t) = \begin{pmatrix} \cos 2t \\ \sin(t/2) \end{pmatrix}$.

(b) $A = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ and $\vec{f}(t) = 0$.

(c) $A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\vec{f}(t) = \begin{pmatrix} \exp(t) \\ t \end{pmatrix}$.

(d) $A = \begin{pmatrix} t^2 & -t \\ t & t^2 \end{pmatrix}$ and $\vec{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution: In each case the matrix A commutes with $B = \int_0^t A dt$. Hence we can see that the matrix of solutions is given by $G = \exp(B)$. In other words, we have the identity $(dG/dt) = A \cdot G$. Now, we put $\vec{v} = G \cdot \vec{u}$ and find the identity $G(d\vec{u}/dt) = \vec{w}$. In other words, we can calculate $v = \int_0^t (G^{-1} \cdot \vec{w})\, dt$.

2. Given the linear ODE

$$\frac{d\vec{v}}{dt} = A(t) \cdot \vec{v}$$

where $A(t)$ is given as below. Assume that $\vec{v_1}$ is the solution with initial value $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\vec{v_2}$ is the solution with initial value $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (**Note: To solve the following exercises, you do *not* need to solve the equations!**)

(a)

$$\begin{pmatrix} 0 & \cos t \\ -\cos t & 0 \end{pmatrix}$$

What can you say about the lengths of the two vectors a function of t?

Solution: Since the matrix is skew-symmetric, one can show that it preserves the langth of *every* initial vector $\vec{v}_0$.

(b)

$$\begin{pmatrix} t & \cos t \\ -\cos t & t \end{pmatrix}$$

What can you say about the angle between the two vectors a function of t?

Solution: Since the matrix can be written as $t(1)_2 \cos tI$, one can see that expontial $\exp(tA)$ can be seen as a rotation followed by a scaling. It follows that angles are preserev=ed.

(c)

$$\begin{pmatrix} 1 & t \\ -t & t^2 \end{pmatrix}$$

What can you say about the length of the cross-product of the two vectors as a function of t?

Solution: As the trace of A is $1 + t^2$, one can show thave $\exp(A)$ has determinant $\exp(t + t^3/2)$. This is the same as the cross-product of its component vetors.