Solutions to Assignment 2

1. Solve the linear ordinary differential equations with constant coefficients

$$\frac{d\vec{v}}{dt} = A \cdot \vec{v}$$

for each of the following matrices A by calculating $\exp(tA)$.

(a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Solution: The characteristic polynomial of this matrix is $T(T-1)+1$ which is the same as $(T-1/2)+3/4$. Following the notes, we put

$$J = \frac{\left(A - \frac{1}{2}\mathbf{1}_2\right)}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$$

We then have $J^2 = -\mathbf{1}_2$. Hence, if we put

$$G = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, J \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{pmatrix}$$

then $J \cdot G = G \cdot I$ where

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Following the notes once again, we see that

$$\exp(tA) = \exp(t/2) G \cdot R\left(t \cdot \frac{\sqrt{3}}{2}\right) \cdot G^{-1}$$

where

$$R(u) = \exp(uI) = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$$

To complete the calculation (up to matrix multiplication!) we note that

$$G^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$$

(b) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Solution: The characteristic polynomial of A is $(T-1)^2+1$. We follow the same technique as above. We note that

$$(A - \mathbf{1}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = I$$

Now, as above we define

$$R(t) = \exp(tI) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Since $I = (A - \mathbf{1}_2)$ we have $R(t) = \exp(tA) \cdot \exp(-t\mathbf{1}_2)$. In other words

$$\exp(tA) = \exp(t) \cdot R(t)$$

(c) $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

Solution: The characteristic polynomial of A is $(T-1)(T+1)+1$ or equivalently T^2. In other words, this is a nilpotent matrix. By inspection we see that the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in the image of A. It follows that if we put

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$A \cdot G = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = G \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Using the fact that

$$\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

It follows that

$$\exp(tA) = G \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot G^{-1}$$

where we calculate

$$G^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

(d) $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

Solution: The characteristing polynomial of A is $T(T+1)-1)$, or equivalently $(T+1/2)^2 - 5/4$. We calculate

$$A - \frac{-1+\sqrt{5}}{2}\mathbf{1}_2 = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{pmatrix}$$

The vector $\vec{v_1} = \begin{pmatrix} -2 \\ 1-\sqrt{5} \end{pmatrix}$ is sent to 0 by the above matrix. Similarly, we check that $\vec{v_2} = \begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix}$ is mapped to 0 by $A - (-1-\sqrt{5})/2\mathbf{1}_2$. Following the method outlined in the notes, we then put

$$G = \begin{pmatrix} -2 & -2 \\ 1-\sqrt{5} & 1+\sqrt{5} \end{pmatrix}$$

and check that

$$A \cdot G = \begin{pmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ -3+\sqrt{5} & -3-\sqrt{5} \end{pmatrix} = G \cdot \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & 0 \\ 0 & \frac{-1-\sqrt{5}}{2} \end{pmatrix}$$

It follows that

$$\exp(tA) = G \cdot \begin{pmatrix} \exp\left(t\frac{-1+\sqrt{5}}{2}\right) & 0 \\ 0 & \exp\left(t\frac{-1-\sqrt{5}}{2}\right) \end{pmatrix} \cdot G^{-1}$$

where

$$G^{-1} = \frac{-1}{4\sqrt{5}} \begin{pmatrix} 1+\sqrt{5} & 2 \\ -1-\sqrt{5} & -2 \end{pmatrix}$$

(e) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Solution: The characteristic polynomial of A is $T(T-1)-1$, or equivalently $(T-1/2)^2 - 5/4$. We calculate

$$A - \frac{1+\sqrt{5}}{2}\mathbf{1}_2 = \frac{1}{2}\begin{pmatrix} -1-\sqrt{5} & 2 \\ 2 & 1-\sqrt{5} \end{pmatrix}$$

The vector $\vec{v_1} = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}$ is sent to 0 by the above matrix. Similarly, we check that $\vec{v_2} = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$ is mapped to 0 by $A - (1-\sqrt{5})/2\mathbf{1}_2$. Following the method outlined in the notes, we then put

$$G = \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix}$$

and check that

$$A \cdot G = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 3+\sqrt{5} & 3-\sqrt{5} \end{pmatrix} = G \cdot \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

It follows that

$$\exp(tA) = G \cdot \begin{pmatrix} \exp\left(t\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & \exp\left(t\frac{1-\sqrt{5}}{2}\right) \end{pmatrix} \cdot G^{-1}$$

where

$$G^{-1} = \frac{-1}{4\sqrt{5}} \begin{pmatrix} 1-\sqrt{5} & -2 \\ -1-\sqrt{5} & 2 \end{pmatrix}$$