Differential Equations for Scientists

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Examples

We look at some examples from science which can be formulated as ODE.

Infectious disease

There are three populations: the healthy population H, the diseased population D and the susceptible population S. (The diseases does not create immunity and is not dangerous enough to result in death.)

- Some fraction αD of the diseased population becomes cured and returns to being healthy.
- Some fraction βS of the susceptible population becomes diseased and some fraction γS of it returns to being healthy.
- The healthy population has had no contact with the disease but when it does interact with the diseased population it becomes susceptible. We can consider this interactive term as δHD.

Thus, the differential equations are:

$$\frac{dD}{dt} = -\alpha D + \beta S$$
$$\frac{dS}{dt} = -\beta S - \gamma S + \delta HD$$
$$\frac{dH}{dt} = -\delta HD + \alpha D + \gamma S$$

Chain

A chain is hanging from a table and slipping down. The friction on the table is proportional to the length T of the chain on the table; the gravitational force pulling the chain down is proportional to the length H of the chain that is hanging down. The combined force is being applied to the full chain by Newton's law. The combined acceleration is $\alpha H - \beta T$. We have $H + T = L$ since the length L of the chain is not changing. So the equation of motion is $d^2H/dt^2 = \alpha H - \beta T$. We can write this as

$$\frac{dH}{dt} = V$$
$$\frac{dV}{dt} = \alpha H - \beta(L - H)$$

Equations of the flow curves

The flow in the plane is given by the field $\vec{v}(x, y) = (f(x, y), g(x, y))$. What are the equations of the flow lines?

Note that the differential equation for the flow is given by:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

However, we do not need to *solve* this to obtain the equation of the curves along which the flow occurs! The vector $\vec{v}(x, y)$ is *tangent* to the curve. So if the curve is in the form $y = \Phi(x)$, then (dy/dx) is the *slope* of the line along this vector. In other words, $\Phi(x)$ is the solution of the differential equation $(dy/dx) = f(x, y)/g(x, y)$ (wherever $g(x, y) \neq 0$).

Of course, where $f(x, y) \neq 0$, then we can try to write the curve as $x = \Psi(y)$ which is a solution of $(dx/dy) = g(x, y)/f(x, y)$.

The points (x, y) where $\vec{v}(x, y) = 0$ are called *critical* points of the flow since the flow curve is not defined at those points.

Further, we can *also* write the equations of curves that are "wavefronts" of the flow. These are curves that are *perpendicular* to the flow at each point. Since the slope of any line perpendicular to a line with slope m is given by $-1/m$, the differential equation for these curves is given by $(dy/dx) = -f(x, y)/g(x, y)$.

Brine tank

A tank contains a volume A of brine (salty water) which contains a quantity B of salt dissolved in it. Pure water is added to it at a constant rate α while stirring. As a result of stirring, water spills out of the tank at a constant rate β. How do A and B evolve with time?

Clearly $(dA/dt) = \alpha - \beta$. Now, B/A is the fraction of salt in a unit quantity of water. Thus $\beta(B/A)$ is the rate at which salt is decreasing; salt is not increasing since pure water is flowing in. So $(dB/dt) = \beta(B/A)$. So the equations are:

$$\frac{dA}{dt} = \alpha - \beta$$
$$\frac{dB}{dt} = -\beta(B/A)$$

Drama

In a drama:

- The hero is running towards the villain.
- The villain is running towards the mother of the hero to get a hostage.
- The mother of the hero is running toward the hero to protect her "child".

Assuming that they are all running on a plane at their own fixed speeds, write a differential equation for their trajectories.

Suppose that $\vec{h}$ is the position of the hero, $\vec{v}$ is the position of the villain and $\vec{m}$ is the position of the mother; let α, β and γ be their respective speeds. Then, we have

$$\frac{d\vec{h}}{dt} = \alpha \frac{\vec{v} - \vec{h}}{||\vec{v} - \vec{h}||}$$
$$\frac{d\vec{v}}{dt} = \beta \frac{\vec{m} - \vec{v}}{||\vec{m} - \vec{v}||}$$
$$\frac{d\vec{m}}{dt} = \gamma \frac{\vec{h} - \vec{m}}{||\vec{h} - \vec{m}||}$$