Differential Equations for Scientists

Kapil Hari Paranjape

What is a differential Equation

Before we try to understand *differential* equations, we should first try to understand what one means by an equation! In high school, one learned to solve (a system of) linear equations. For example, the problem of finding x, y and z so that

$$\begin{aligned} 2x + 3y + 5z &= 22 \\ 7x - 5y - 3z &= 10 \end{aligned}$$

is a problem of solving linear equations. The more fancy way of stating this problem is to write it as a matrix identity

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & -5 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 22 \\ 10 \end{pmatrix}$$

or equivalently $M \cdot X = B$ with appropriate values for M, X and B. A solution can be found[1] by using row-reduction of the 2×3 matrix M; this is a column vector X_0 of length 3 such that $M \cdot X_0 = B$. Further, if we have a column vector Y_0 so that $M \cdot Y_0 = 0$, then we see easily that $M \cdot (X_0 + Y_0) = B$ is *another* solution of the linear equation.

The problem of solving for X so that the identity $M \cdot X = B$ holds, where M and B are *known* is the problem of solving linear equations. This equation is called the *inhomogeneous* equation and the equation $M \cdot Y = 0$ is called the *homogeneous* equation. A similar situation will arise in differential equations and so the terminology *inhomogeneous* and *homogeneous* will re-appear in that more general setting.

A different problem studied in high school was that of solving quadratic equations. For example, the problem of finding an x so that $x^2 - 5x + 4 = 0$. This is solved by the method of "completing the square" which gives the *two* solutions $x = 1$ and $x = 4$. While this *looks* like a *homogeneous* problem, it is not really so! We will see this more clearly once we understand the terminology better.

A more complex problem is that of solving a quadratic equation in *two* variables. For example, the problem of finding x and y such that $x^2 + y^2 = 1$. One “classical” solution is given in the parametric form

$$x = \frac{1-t^2}{1+t^2}$$
$$y = \frac{2t}{1+t^2}$$

A more “traditional” solution is $(x, y)(t) = (\cos(t), \sin(t))$. (Question: What is the difference between the adjectives “classical” and “traditional”!😄) We note that the first solution does not makes sense for *all* values of t since $t^2 + 1$ could be 0! It *also* misses the solution $(x, y) = (-1, 0)$. On the other hand, the second solution (may) look like “cheating” since it introduces two “new” functions sine and cosine which are just meant to solve this particular equation! The important thing to note is that the solution to an equation *can* be a (collection of) function of one (or more) parameters. You will see many such examples in your course on Curves and Surfaces.

Looking at the above examples and others like them (and being mathematically minded) we quickly generalise and say that the problem of solving equations can be stated as finding $x_1, \ldots, x_q$ which satisfy the identities

$$\begin{aligned} f_1(x_1, \ldots, x_q) &= 0 \\ \vdots \quad & \vdots \quad \vdots \\ f_p(x_1, \ldots, x_q) &= 0 \end{aligned}$$

where $f_1, \ldots, f_p$ are *given* functions of q variables. What kinds of functions? If the functions are all linear functions, then we say these are linear equations, if the functions are quadratic, we say that this is a system of quadratic equations? Of course, we have learned about much more general functions than polynomial functions so the scope is literally limitless! For example, we could ask for a solution of the equations

$$\begin{aligned} \cos(x) + \sin(y) - 3 &= 0 \\ \exp(x) - \log(y) - 10 &= 0 \end{aligned}$$

(No need to challenge yourself by trying to solve this equation. It will not come in the exam!😄)

Now that we have understood what an equation is, let us try to understand what a differential equation is. It is an equation of the same type as before with the *added* condition that the derivatives (or differentials) of the variables can *also* appear in the equation in place of some of the variables. Since you have all studied derivatives properly, you will ask: "Derivatives with respect to what?" The somewhat "correct" answer is: "Derivatives with respect to the parameter." Still confused you ask: "Which parameter?" Answer: "The parameters of the solution." At this point it looks like the definition is circular and you might start feeling like we are following $(\cos(t), \sin(t))$ for too long!😄 So you will have to take it on faith that a general definition of this type can actually be given in a way that makes sense. However, defining the "abstract" notion of a *differential* which can take the place of a derivative, and so on will take us too far afield.

Let us therefore do as our ancestors did and work with examples. (Just as we did for equations above.) The simplest equation we can write using a derivative is $(dy/dx) = f(x)$ for some "simple" function f. This is nice because we already know how to solve this equation! The solution is given by $y = F(x)$ where $F(x) = \int f(x)dx$, where the latter is the (indefinite) integral of f with respect to x; this is just the fundamental theorem of calculus! So, we already learned to solve differential equations like:

- $(dy/dx) = x^2$ with solution $y = x^3/2 + c$, for a constant c,
- $(dy/dx) = \sin(x)$ with solution $y = -\cos(x) + c$, for a constant c,

… and so on. (It is good that in the very first class in a Mathematics course you have learned to solve infinitely many problems, so savour the moment!😄)

Still, the above kind of equation that *already* defines y explicitly as a function of x is not very different from an equation of the type $y^2 = x^3 + 1$ which can be solved by "doing something" with *only* the right-hand-side. A much more interesting type of equation is $(dy/dx) = f(y)$. It may seem perplexing that this is more complex as *all* that we did is to replace x by y on the right-hand-side. Indeed, if $f(y)$ is never zero, then we can write (by inverse function theorem!) $dx/dy = 1/f(y)$ and now solve for x as a function of y. By the inverse function theorem, the inverse of this function, is the solution to our problem. For example:

- $dy/dx = y$ gives $dx/dy = 1/y$ (for $y \neq 0$); which has the solution $x = \log(y) + c$. Inverting this gives $y = \exp(-c)\exp(x)$.
- $dy/dx = y^2$ gives $dx/dy = 1/y^2$; which has the solution $x = -1/y + c$. Inverting this gives $y = -1/(x + c)$.

… and so on. To make the problem significantly more difficulty, we need to do one of the following:

- Replace y by a *vector*-valued function of x and f by a vector valued function. Equivalently, we can consider a *system* of equations

$$\begin{aligned} \frac{dy_1}{dx} &= f_1(y_1, \ldots, y_p) \\ \vdots \quad & \vdots \quad \vdots \\ \frac{dy_p}{dx} &= f_p(y_1, \ldots, y_p) \end{aligned}$$

- We can make f a function of x *and* y.

If we want to make our work even more difficult, we can *combine* these two approaches and consider systems of equations of the type:

$$\begin{aligned} \frac{dy_1}{dx} &= f_1(x, y_1, \ldots, y_p) \\ \vdots \quad & \vdots \quad \vdots \\ \frac{dy_p}{dx} &= f_p(x, y_1, \ldots, y_p) \end{aligned}$$

This kind of equation will be the main focus of our study in the first part of this course.

Why do want to study differential equations?

The above example *may* be good enough to motivate mathematicians. Why would scientists want to learn differential equations?

A wonderful explanation is given in the book “The Character of Physical Law” by Richard Feynman. In it he explains why it is natural for physical laws to be expressed in the form of differential equations.

Rather than repeat his explanation here, let us look at it in our own way. The job of science is to make predictions about a changing universe. The fundamental way we keep track of change is by measuring some physical quantity at different times. Once we have a clock (the most fundamental physical instrument!), we can assign a (real) number to each time epoch. At each such epoch we also measure various physical quantities; these too are given as real numbers. In other words, the changing physical quantity is represented by a function $f : \mathbb{R} \to \mathbb{R}$ which gives the value $f(t)$ of the physical quantity at time t. More generally, we may keep track of a number of such physical quantities and so we have a function $f : \mathbb{R} \to \mathbb{R}^n$; a vector-valued function in the mathematical sense.

Physical laws typically do not *directly* provide a formula for this function f. This is because, at the very least, different people have different clocks and so this function is not "universal". Instead, there are two kinds of laws that we see:

- The physical quantities $f_1, \ldots, f_p$ depend on each other in a particular way. Mathematically, this is formulated as a function $\Psi(f_1, \ldots, f_p)$ remaining unchanged with time.
- The physical quantities f changes with time as a function Φ of its *current* value. In other words, an identity of the type $(df/dt) = \Phi(f)$ holds.

The first form of a physical law as above is an "conservation" law; it says that something is invariant with time. The second form is an "evolution" or "dynamical" law. It says that nature exhibits a "feedback" mechanism; it says that the way physical quantities change depend on on their current values.

It is clear that the second law is in the form of a differential equation. Since some physical quantities are derivatives of other physical quantities (for example, acceleration is the derivative of velocity), the first type of law is often can also be seen as a general form of differential equation. However, there is a more intricate way in which the two types of laws are related which was discovered by Emmy Noether (one of the greatest mathematicians of the previous century). The evolution law can be see as an "infinitesimal" group action; the conservation laws are given as functions that are invariant under this group action. A simple example is as follows.

Suppose that the evolution equation is given in the form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

In other words, $\Phi((x, y)) = (-y, x)$. One can then check

$$\frac{d}{dt}\left(x^2 + y^2\right) = 2\left(x \cdot -y + y \cdot x\right) = 0$$

It follows that the function $\Psi(x, y) = x^2 + y^2$ is a conserved quantity.

We will see more examples of this type during the course.

How to visualise Differential equations

As seen above, in a lot of cases, an ordinary differential equation takes the form $(d\vec{x}/dt = \vec{f}(t, \vec{x})$; in fact, in may cases, $\vec{f}$ does not depend on t either.

Here $\vec{x}$ denotes the n-tuple of physical quantities that we want to see the evolution of. This equation then expresses the dynamical law that describes this evolution. Physically, the n-tuple $\vec{x}$ need not really be a vector and all its entries need not be of the same type. However, when we study differential equations *mathematically* this need not concern us! We can think of $\vec{x}$ as a point in n-dimensional space and $\vec{f}(t, \vec{x})$ as a vector at *that* point. The above equation then describes a (time-dependent) "flow" in n-dimensional space.

In other words, a *solution* of the above equation takes the form $\vec{x}(t) = F(t, \vec{x}_0)$ which represents how a "drop of ink" which starts at $\vec{x}_0$ will be seen to travel if its velocity at all times is described by the above equation. Mathematically, this means $\vec{x}(0) = \vec{x}_0$ and $(d(F(t, \vec{x}_0)/dt) = \vec{f}(t, F(t, \vec{x}_0))$; we will explain this later as an "initial value problem" (IVP).

So, an ordinary differential equation can be visualised as a equation describing the *velocity* at time t and position $\vec{x}$ of a flow that is happening in n-dimensional space. So $\vec{x}_1 = F(t_1, \vec{x}_0)$ is the position at time t_1 of the point which started at $\vec{x}_0$.

Note that time is *also* a physical quantity so our vector of physical quantities can *include* time by putting $\vec{y} = (t, \vec{x})$ and then the differential equation becomes $(d\vec{y}/dt) = G(\vec{y})$ where $G((t, \vec{x})) = (1, F(t, \vec{x}))$. Thus, mathematically speaking, a time-dependent equation can be also seen as a time-independent equation in one larger dimension.

If the flow is time-*independent*, then it is clear that if we track the point at $\vec{x}_1$ (as above), then it passes through a point t_1 units of time *before* the point at $\vec{x}_0$ passes through the same point. In other words, we see that we have an equation:

$$F(t - t_1, \vec{x}_1) = F(t, \vec{x})_0)$$

Substituting for $\vec{x}_1$ and noting that t, t_1 and $\vec{x}_0$ are “arbitrary”, we obtain

$$F(t - s, F(s, \vec{x})) = F(t, \vec{x})$$

With some simple group theory we can see this a little better. Suppose $\mathcal{G}$ denotes the group of all differentiable maps from n-dimensional space to itself whose inverse exists and is differentiable. Then for each time t, we have an element $\Phi(t)$ of $\mathcal{G}$ which describes the “snapshot” of the flow at time t; we then have $\Phi(t)(\vec{x})$ as the position at time t of the point that started at $\vec{x}$. In other words $\Phi(t)(\vec{x}) = F(t, \vec{x})$. Writing the above equation in terms of Φ we see that

$$\Phi(t - s) \circ \Phi(s)(\vec{x}) = \Phi(t)(\vec{x})$$

In other words, $\Phi(t - s) \circ \Phi(s) = \Phi(t)$; the map $\Phi : \mathbb{R} \to \mathcal{G}$ is a group homomorphism!

In summary, an ordinary differential equation can be seen as describing the velocity vector field of a flow in n-dimensional space. When the flow is time-independent, we can see such a flow as the “infinitesimal” version of a one-parameter group of transformations of the n-dimensional space to itself.

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1. This is a good point for students to go back and revise the solutions of linear equations by the row-reduction method.↩