Solutions to Quiz 9

1. For each of the following give an example or say that no such example is possible (do *exactly* 5):

(1 mark) (a) A sequence that contains no convergent subsequence.

> **Solution:** The sequence $(a_n) = (n)$.

(1 mark) (b) A sequence (a_n) so that $\sum_n |a_n|$ converges but $\sum_n |a_n|^2$ does not converge.

> **Solution:** There is no such sequence. Since $\sum_n |a_n|$ converges, $|a_n|$ converges to 0 as n goes to infinity. So there is an N so that $|a_n| < 1$ for $n \geq N$. It follows that
> $$\sum_{n \geq N} |a_n|^2 < \sum_{n \geq N} |a_n|$$
> Thus, $\sum_n |a_n|^2$ converges.

(1 mark) (c) A linear functional which is *not* continuous.

> **Solution:** Consider the functional $f : \mathbb{C}^\infty \to \mathbb{C}$ given by $f((a_1, \ldots, a_n, 0, \ldots)) = \sum_{k=1}^{n} k a_k$. This is not continuous for $\|\cdot\|_\infty$ norm.

(1 mark) (d) A linear operator which is *not* open.

> **Solution:** The operator that sends every vector to 0 is not open since its image is $\{0\}$ which does not contain an open neighbourhood of 0.

(1 mark) (e) An linear operator which is 1-1 and onto but not invertible.

> **Solution:** Consider the operator $T : \mathbb{C}^\infty \to \mathbb{C}^\infty$ given by $(a_1, a_2, \ldots) \mapsto (a_1, a_2/2, \ldots)$. In other words the sequence (a_n) is mapped to the sequence (a_n/n). The inverse operator $(a_n) \mapsto (na_n)$ is *well-defined*. However, when we think of $\mathbb{C}^\infty$ with the $\|\cdot\|_\infty$ norm then T is a bounded operator but T^{-1} not a bounded operator.

(1 mark) (f) A bounded linear operator whose spectrum is empty.

> **Solution:** We have proved that every operator has a non-empty closed spectrum. So there is no such operator. Sometimes, by convention, we can consider the operator $0 : \{0\} \to \{0\}$ to have empty spectrum since it is automatically invertible!

(1 mark) (g) A linear functional $f : \mathbb{C}^\infty \to \mathbb{C}$ which is continuous with ℓ_2 norm but there is no $v \in \mathbb{C}^\infty$ such that $f(w) = \langle w, v \rangle$.

> **Solution:** This appears to contradict Riesz representation theorem but $\mathbb{C}^\infty$ is *not* complete with the ℓ_2 norm! So we can take the functional $f(w) = \langle w, v \rangle$ where v is in ℓ_2 but *not* in $\mathbb{C}^\infty$; for example $v = (1, 1/2, 1/3, \dots)$. This is continuous but there is no v *in* $\mathbb{C}^\infty$ which works.

(1 mark) (h) An linear operator on a Hilbert space which is not normal.

> **Solution:** We know that normal compact operators are diagonalisable. Hence, a finite-rank operator which is not diagonalisable is not normal. For example,
>
> $$(a_1, a_2, \dots) \mapsto (0, a_1, 0, 0, \dots)$$
>
> This operator is nilpotent and non-zero, so it is not diagonalisable.

(1 mark) (i) A linear operator on a Hilbert space which preserves the norm but is not unitary.

> **Solution:** In infinite dimensions, preserving norm *does not* mean that the operator is onto. So, for example, the right-shift operator $R : \ell_2 \to \ell_2$ is norm preserving but its image lies in a proper closed subspace. Hence, it does not have an inverse and so it is not unitary.

(1 mark) (j) A linear operator on a Hilbert space that does not have a Hilbert basis of eigenvectors.

> **Solution:** Any of the above two examples works! In other words, the right-shift operator does not have a Hilbert basis consisting of eigenvectors. Similarly, a non-zero nilpotent operator of finite rank does not have a basis consisting of eigenvectors.