Solutions to Quiz 7

1. Give an example for each of the following or indicate that no example is possible:

(1 mark) (a) A compact operator which is not of finite rank.

(1 mark) (b) A Fredholm operator that is not onto.

(1 mark) (c) An operator that is not compact.

(1 mark) (d) An operator that is not Fredholm.

(1 mark) (e) An operator that is Fredholm and compact.

Solution: The operator $S : \ell_1 \to \ell_1$ given by $(a_n) \mapsto (a_n/2^n)$ is a nuclear operator and so it is compact. It is not of finite rank.

The right-shift operator $R : \ell_1 \to \ell_1$ given by

$$(a_1, a_2, \dots) \mapsto (0, a_1, a_2, \dots)$$

is not onto. It has image equal to the closed subspace consisting of (a_n) where $a_1 = 0$. Its kernel is $\{0\}$. Since its kernel is finite dimensional, its image is closed and of finite codimension, so it is a Fredholm operator.

The identity operator $\mathbf{1} : \ell_1 \to \ell_1$ is not compact since the unit ball in ℓ_1 is not compact.

The operator $0 : \ell_1 \to \ell_1$ which sends all elements to 0 is not Fredholm since the kernel is infinite dimensional.

Any operator $A : F \to G$ where F and G are *finite* dimensional spaces is a Fredholm operator and also compact. It is not difficult to prove the converse that these are the only compact Fredholm operators.