Compact Operators

An operator $T : V \to W$ is called a *compact* operator if the image of a bounded sequence contains a convergent subsequence.

Consider the subset set $K = \overline{T(B(0,1)}$ in W. Given a sequence y_n in K, each y_n lies in the closure of $T(B(0,1)$. So we have an x_n in $B(0,1)$ so that $\|Tx_n - y_n\| < 1/n$. Now, by the compactness of T, there is a subsequence $n_1 < n_2 < \cdots$ so that Tx_{n_k} converges in W; let y be its limit. It follows that y_{n_k} also converges to y. Thus, we see that any sequence in K has a convergent subsequence. In other words, K is *sequentially compact*. Since K is contained in a metric space, this is the same as saying that K is compact.

In other words, an operator $T : V \to W$ is compact if and only if $\overline{T(B(0,1))}$ is compact. Since a compact set is closed and bounded, so it follows that T is a bounded operator.

Divergent sequences

In order to check the consequences of compactness we need to have a method to create sequences that *do not* contain a convergent subsequence! Let U be a *closed proper* subspace of V. Let v be in V and not in U. Define a linear functional $f : U + \mathbb{C} \cdot u \to \mathbb{C}$ by $f(u + z \cdot v) = z$.

Exercise: Check that f is continuous. (Hint: Use the fact that U is closed.)

Exercise: Show that there is a continuous linear functional $g : V \to \mathbb{C}$ which is identically 0 on U so that $g(v) \neq 0$. (Hint: Use the Hahn-Banach theorem.)

Scaling g if necessary, let us assume that $\|g\| = 1$. So we have a linear functional that is 0 on U and has norm 1.

Exercise: Show that there is a *unit* vector w so that $|g(w)| > 1/2$. (Hint: Use the definition of $\|g\|$.)

For any u in U, we have $g(u + w) = g(w)$. It follows (using $\|g\| = 1$) that

$$1/2 < |g(u + w)| \leq \|g\|\|u + w\| = \|u + w\|$$

In other words, we have $d(w, U) > 1/2$ (where $d(w, U) = \inf\{\|w - u\| : u \in U\}$).

Let $0 \neq V_1 \subsetneq V_2 \cdots$ be an *increasing* sequence of *closed* subspaces of V. Applying the above to $V_n \subsetneq V_{n+1}$ we can find a sequence x_{i+1} in $V_{i+1} \setminus V_i$ so that $d(x_{i+1}, V_i) > 1/2$.

Exercise: Note that x_i *cannot* contain a convergent subsequence. (Hint: Prove that $\|x_i - x_j\| > 1/2$ whenever $i \neq j$.)

Thus, given an increasing sequence of closed subspaces V_n, we can find a *unit* vector x_n in each V_n, so that the sequence (x_n) does not contain a convergent subsequence.

Now, as seen earlier, a finite dimensional subspace of a normed linear spaces is *automatically* closed. Hence, if V is infinite dimensional, we can take V_n to be of dimension n. It follows that (x_n) is does not contain a convergent subsequence.

Exercise: Show that an infinite dimensional normed linear spaces is *not* locally compact. Equivalently, show that the closed unit ball $\overline{B(0,1)}$ in an infinite dimensional normed linear space is not compact. Equivalently, show that the identity operator $\mathbf{1} : V \to V$ for such a space is not a compact operator.

In other words, the identity operator on a space is compact *only if* the space is finite dimensional. Since the closed unit ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ is compact, we see that this necessary condition is also sufficient.

Arzela-Ascoli Theorem

Given a compact operator $T : V \to W$, we claim that its dual operator $T^t : W^* \to V^*$ is also compact. (Recall that $T^t f$ is defined by $(T^t f)(v) = f(Tv)$.) In order to prove this we need to show that if (f_n) is a bounded sequence of elements of W^* then $T^t f_n$ contains a convergent subsequence. Let us assume that $\|f_n\| \leq C$ for all n.

We consider (f_n) as a sequence of functions on the compact set $K = \overline{T(B(0,1))}$. From the condition that f_n's are linear functionals with norms that are uniformly bounded by C, we see that

$$|f_n(w) - f_n(w')| \leq C\|w - w'\|$$

It follows that given $\epsilon > 0$, we can take $\delta = \epsilon/2C$ to ensure that if $d_K(w, w') < \delta$, then $|f_n(w) - f_n(w') < \epsilon$ for *all* n. (Here onward, to simplify notation, we use $d_K(w, w') = \|w - w'\|$.) Such a collection of continuous functions for which, given $\epsilon > 0$, the *same* $\delta > 0$ "works" for all the functions in the collection is called an *equicontinuous* collection of functions.

We need to prove that an equicontinuous sequence of functions on a compact metric space K contains a uniformly convergent subsequence. This is the theorem of Arzela-Ascoli, which we will now prove.

Exercise: Show that K contains a countable dense set. (Hint: For each n let $\{x_{m,n}\}_{m=1}^{n_r}$ be the finite set of points so that $\cup_m B(x_{m,n}, 1/n)$ cover K and show that $\{x_{m,n}\}_{m=1,n=1}^{n_r,\infty}$ is dense in K.)

To simplify notation, let $\{y_p\}_p$ denote a countable dense set in K. Since the sequence f_n is *uniformly* bounded on K, the sequence $(f_n(y_1))$ is a bounded collection of real numbers. Hence we can find a subsequence $(n_{1,k})$ so that $(f_{n_{1,k}}(y_1))_k$ is convergent. We now consider the bounded sequence $(f_{n_{1,k}}(y_2))$ of real numbers. We can find a subsequence $(n_{2,k})$ of $(n_{1,k})$ so that $(f_{n_{2,k}})(y_2)$ is convergent. Note that $(f_{n_{2,k}}(y_1))$ is also convergent since $n_{2,k}$ is a subsequence of $n_{1,k}$.

Repeating this process, we can find subsequences (which are nested, i.e. each a subsequence of the previous one) $(n_{p,k})_k$ so that $(f_{n_{q,k}}(y_p))_k$ is convergent for all $p \leq q$.

We then put $m_k = n_{k,k}$ (the "diagonal trick").

Exercise: Check that $(f_{m_k}(y_p))$ is convergent for all p.

We thus see that we have found a subsequence $(g_k = f_{m_k})$ of (f_n) which converges on the dense set $\{y_p\}_p$ in K. We now show that this converges uniformly on all of K.

Given $\epsilon > 0$, for each p we choose $N_{\epsilon,p}$ so that $|g_m(y_p) - g_n(y_p)| < \epsilon/3$ for all $m, n \geq N_{\epsilon,p}$.

Secondly, using equicontinuity of the sequence, we choose δ_ϵ so that $|g_m(y) - g_m(y')| < \epsilon/3$ for all $y, y' \in K$ so that $d_K(y, y') < \delta_\epsilon$.

By the density of $\{y_p\}$, the union of $B(y_p, \delta_\epsilon)$ over all p covers K. By the compactness of K, there is a finite collection $p_1, \ldots, p_{r_\epsilon}$ so that that $B(y_{p_t}, \delta_\epsilon)$ cover all of K as t varies from 1 to p_{r_ϵ}.

We take $N \geq N_{\epsilon,p_t}$ for all t in 1 to p_{r_ϵ}.

Exercise: Check that $|g_n(y) - g_m(y)\| < \epsilon$ for all $n, m \geq N$. (Hint: Combine the above inequalities.)

In other words, the sequence (g_n) converges uniformly in the space of continuous functions on K. The limit is thus continuous as required. We note that $T^t(g_n)$ is just $g_n \circ T$. It follows that $h_n = T^t(g_n)$ converges to a continuous function $\tilde{h}$ on $B(0, 1)$. We now define

$$h(v) = \begin{cases} 0 & v = 0 \\ 2\|v\|\tilde{h}(v/2\|v\|) & v \neq 0 \end{cases}$$

Exercise: Show that h is equal to $\tilde{h}$ on $B(0, 1)$ and that h is a continuous linear functional on V and that h_n converge to h in the norm topology on V^*. (Hint: Repeated use of the linearity of h_n and continuity of the operations of addition and scalar multiplication on a normed linear space.)

This completes the proof that T^t is a compact operator.

In passing, let us note that if V is a linear space, then V is a subspace of V^{**} in a natural way.

Exercise: Given a linear space V define $\lambda_V : V \to V^{**}$ by defining, for a vector v in V and a linear functional f in V^*, the value $\lambda_V(v)(f) = f(v)$. Show that this is λ_V is linear and is continuous if V is a normed linear space.

Now, if T^t is a compact operator, then, as proved above, so is $(T^t)^t : V^{**} \to W^{**}$.

Exercise: Check that $(T^t)^t \circ \lambda_V$ is just $\lambda_W \circ T$.

It follows that T is compact whenever T^t is.

Finiteness of non-zero eigenspaces

Let z be a non-zero complex number and $T : V \to V$ a compact operator. We wish to show that the subspace N_z of V consisting of eigenvectors for eigenvalue z is finite dimensional. (In what follows we can allow for the case where there are no such eigenvectors, in other words, where z is *not* an eigenvalue.)

Exercise: Show that v is an eigenvector with eigenvalue 1 for $(1/z) \cdot T$ if and only if v is an eigenvector with eigenvalue z for T.

Secondly, we note:

Exercise: Show that $(1/z) \cdot T$ is compact if T is compact.

Thus, we can limit ourselves to studying the subspace of eigenvectors for eigenvalue 1.

Suppose this space is *not* finite dimensional. We will then show that T is *not* a compact operator. To do so, let us use the results of a previous subsection to construct a sequence of *unit* vectors x_n which are eigenvectors of T with eigenvalue 1 such that $\|x_n - x_m\| > 1/2$ whenever $n \neq m$. Since $Tx_n = x_n$, it follows that $\|Tx_n - Tx_m\| > 1/2$. As a consequence, the sequence $\{Tx_n\}$ does not contain a convergent subsequence. Hence T is not compact.

We conclude that the spaces of eigenvectors of a compact operator associated with a (fixed) non-zero eigenvalue are finite dimensional.

Applying this to T^t, which is also a compact operator by the previous subsection, we conclude the same for this operator as well.

Exercise: Show that $T^t(f) = z \cdot f$ if and only if f is identically 0 on the image of $T - z \cdot \mathbf{1}$.

It follows that given any $z \neq 0$ there are finitely many linearly independent continuous linear functionals $f_1, \ldots, f_k$ so that *any* continuous linear functional that vanishes on the image of $T - z \cdot \mathbf{1}$ is a linear combination of $f_1, \ldots, f_k$.

Exercise: Show that the image of $T - z \cdot \mathbf{1}$ is the intersection of the zero sets of *each* f_n.

It follows that the image of $T - z \cdot \mathbf{1}$ is closed.

Exercise: Given a finite linearly independent collection of linear functionals $f_1, \ldots, f_k$ on a vector space V, show that there are vectors $v_1, \ldots, v_k$ so that $f_i(v_j) = \delta_{i,j}$. (In other words, $f_i(v_i) = 1$ and $f_i(v_j) = 0$ if $i \neq j$.)

Exercise: With notation as above, show that V is the direct sum of the closed subspaces $I_z = \text{Image}(T - z \cdot \mathbf{1})$ and the finite-dimensional span F of the vectors $v_1, \ldots, v_k$.

Let $w_1, \ldots, w_r$ be a basis of the finite-dimensional space $N_z = \text{Ker}(T - z \cdot \mathbf{1})$, which is the space of eigenvectors of T with eigenvalue z. We can define linear functionals $g_1, \ldots, g_r$ on this finite-dimensional space N_z, such that $g_i(w_j) = \delta_{i,j}$. Since N_z is finite-dimensional, these linear functionals are automatically continuous! So, by Hahn-Banach theorem, we can extend g_i to continuous linear functions h_i on V. Let K be the intersection of the kernels of each h_i, i.e.

$$K = \{v \in V : h_i(v) = 0 \text{ for all } i = 1, \ldots, r\}$$

Then K is closed. Moreover,

Exercise: Show that V is the direct sum of K and N_z.

Exercise: Show that $T - z \cdot \mathbf{1}$ gives a one-to-one and onto map from K to $\text{Image}(T - z \cdot \mathbf{1})$. (Hint: The map is 0 on N_z.)

Since both of these spaces are closed subspaces of V, they are Banach spaces if V is a Banach space. It follows that $T - z \cdot \mathbf{1}$ is an invertible map between these spaces in that case. In fact, we can show this more directly (without using the Baire category theorem).

We claim that there is a constant $c > 0$ so that

$$\|(T - z \cdot \mathbf{1})(v)\| \geq c\|v\| \text{ for all } v \in K$$

Suppose to the contrary that there is a sequence of unit vectors v_n in K such that $(T - z\mathbf{1})(v_n)$ goes to 0 as n goes to infinity. By the compactness of T there is a subsequence $\{n_k\}_k$ so that Tv_{n_k} converges to some v in V as k goes to infinity. It follows that

$$zv_{n_k} = Tv_{n_k} - \left((Tv_{n_k} - z\mathbf{1})\,(v_{n_k})\right) \text{ converges to } v \text{ as } k \to \infty$$

Then, by continuity of T, we would get that Tv is the limit of zTv_{n_k} which is zv. In other words, v would be an eigenvector of T with eigenvalue z. Since v_{n_k} lie in K, and K is closed, this would mean that v lies in K and N_z which only consists of 0. On the other hand v_{n_k} are unit vectors and so their limit (by continuity of norm) must be a unit vector. This is a contradiction.

To summarise, we get a "structure" theorem for V and T with respect to a non-zero complex number. Let $N_z = \text{Ker}(T - z \cdot \mathbf{1})$ be the collection of eigenvectors for eigenvalue z (which can consist of 0 if there are no such eigenvectors). Then there is a closed subspace K of V so that V is the direct sum $N_z \oplus K$. On the other "side", there is a finite dimensional space F so that V is a direct sum of F and $I_z = \text{Image}(T - z\mathbf{1})$. Thinking T as a map $N_z \oplus K \to F \oplus I_z$ it has a matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}$$

where $G : K \to I_z$ is an isomorphism. It is worth re-iterating that N_z and F are finite dimensional spaces.

Continuity of index

The index of the map $T - z\mathbf{1}$ is defined as the difference $\dim(F) - \dim(N_z)$. In words, it is the difference between the codimension of its image and the dimension of its kernel. Note that, for an isomorphism, both numbers are 0, so the index is 0.

We now want to show that the index is a continuous function of z. Since the value is an integer and the complex numbers are connected, the index will be *constant*. We know that $T - z\mathbf{1}$ is *invertible* for large values of z (for example if $z > \|T\|$). This shows that the index is 0.

As seen above, given a finite dimensional subspace N of normed linear space V, we can use the Hahn-Banach theorem to produce a *supplementary* closed subspace K; in other words, V is the direct sum of N and K. While proving the Hahn-Banach theorem we have seen that the norm on V is equivalent to the norm on the direct sum given by $\|(n, k)\| = \|n\| + \|k\|$. In other words, we have positive constants $C_1 > 0$ and $C_2 > 0$ so that

$$C_1\|n + k\| \leq \|n\| + \|k\| \leq C_1\|n + k\|$$

for all n in N and k in K. A different proof of the same equivalence can be found when V is a Banach space by using the open mapping theorem.

Also seen in the previous section, that if I is a closed subspace of V so that it is defined by the vanishing of finitely many continuous linear functionals, then there is a finite dimensional supplement F to I in V.

Exercise: Show that a closed subspace I of V is defined as the locus of vanishing of a finite collection of continuous linear functionals if and only if V/I is a finite dimensional vector space.

Again, in this case, we can see that the norm on V is equivalent to the norm on the direct sum given by $\|(p, q)\| = \|p\| + \|q\|$ for all p in F and q in I.

When we have decompositions as above, we can view a continuous linear operator $L : V \to V$ in the "block matrix" form as given above

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A : N \to F$, $B : K \to F$, $C : N \to I$ and $D : K \to I$ are continuous linear operators. Moreover, the usual norm on linear operators $V \to V$ can be seen to be *equivalent* to the norm $\|A\| + \|B\| + \|C\| + \|D\|$. (Note: We can also take *any* other norm on 2×2 matrices and it is equivalent.) In particular, we can note that a linear operator L' which has the form

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

is "close" to L if and only if each of the components is "close" to the corresponding component of L. This idea is critical to the proof that follows.

We will now assume that L has a specific form similar to the one found for $T - z\mathbf{1}$ in the previous subsection. Specifically, let us assume that L has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}$$

where $G : K \to I$ is invertible. Such a map, is called a *Fredholm* operator; it has a finite dimensional kernel N and a finite dimensional co-kernel (supplement F to the image I which is closed) and gives an isomorphism (G) between the supplement K to N and the image I.

Instead of proving continuity (local constancy) of the index for $T - z\mathbf{1}$, we will show that *any* operator "close enough" to a Fredholm operator is also a Fredholm operator and has the same index.

As seen in an earlier section, if $D : K \to I$ is sufficiently close to G, then D is also invertible.

Exercise: Show that if $\|D - G\| < 1/\|G^{-1}\|$, then D is invertible.

Hence, if M is sufficiently close to L and has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then $D : K \to I$ is invertible. We want to use this to compute the index of M and show that it is the same as $\dim(F) - \dim(N)$.

We note that the map $P : F \oplus I \to F \oplus I$ given by the matrix

$$\begin{pmatrix} \mathbf{1}_F & -BD^{-1} \\ 0 & \mathbf{1}_I \end{pmatrix}$$

is an isomorphism. Similarly, the map $Q : N \oplus K \to N \oplus K$ given by the matrix

$$\begin{pmatrix} \mathbf{1}_N & 0 \\ -D^{-1}C & \mathbf{1}_K \end{pmatrix}$$

is also an isomorphism. It follows that the kernel (respectively co-kernel) of M are isomorphic to the kernel (respectively co-kernel) of $PMQ : N \oplus K \to F \oplus I$. We calculate

$$PMQ = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}$$

Since D is an isomorphism we see that the kernel and co-kernel of PMQ can be identified with the kernel and co-kernel of $A - BD^{-1}C : N \to F$.

A more computational (and less "magical") way of seeing this is given below.

In order to understand the image of M, we have to solve $M(n + k) = p + q$. This gives $A(n) + B(k) = p$ and $C(n) + D(k) = q$. Again, using the fact that D is an isomorphism, we can write $k = D^{-1}(q - C(n))$. It follows that $(A - BD^{-1}C)(n) + BD^{-1}(q) = p$. In other words, we need to solve

$$(A - BD^{-1}C)(n) = p - BD^{-1}(q)$$

Note that the map $(p, q) \mapsto (p - BD^{-1}(q), q)$ is an isomorphism from V to itself. Thus, the supplement to the image of $(A - BD^{-1}C)$ in F is also a supplement to the image of M.

In other words, we have shown isomorphisms between the kernel and supplement of M and the kernel and supplement of $A - BD^{-1}C$. The important point is that the latter is a linear map between the *finite* dimensional spaces N and F.

Exercise: Let $A' : N \to F$ be a linear map between finite dimensional spaces N and F. Let N_A be the kernel of A' and F_A be a supplement to the image of A'. Show that $\dim(F_A) - \dim(N_A) = \dim(F) - \dim(N)$. (Hint: Use the rank-nullity theorem in linear algebra!)

To summarise, we have shown that the index of M is the same as the index of L whenever M is close enough to L. This is precisely what we wanted to prove.

Spectrum of a Compact operator

We now study the spectrum of a compact operator $T : V \to V$ when V is a Banach space.

One the important consequence of the results of the previous section is that (for $z \neq 0$) the dimension of the kernel of $T - z\mathbf{1}$ is the same as the dimension of the co-kernel $V/\text{Image}(T - z\mathbf{1})$. In particular, $T - z\mathbf{1}$ is one-to-one if and only if it is onto. Moreover, in this case, we have seen that $T - z\mathbf{1}$ is invertible.

In other words, $z \neq 0$ lies in the spectrum of T if and only if there is a non-zero eigenvector with eigenvalue z.

Exercise: Show that the left-shift (or right-shift) operator is not compact.

As proved earlier, the closed unit ball $\overline{B(0, 1)}$ is compact if and only if V is a finite-dimensional space.

Exercise:: Show that if $T : V \to V$ is a compact operator and V is an infinite dimensional Banach space, then T is not onto. (Hint: If T is compact and onto then show that the closure of the unit ball is compact.)

It follows that the spectrum of T always contains 0 if T is a compact infinite-dimensional operator.

We have seen that the spectrum $\sigma(T)$ is contained in the set $\{z : |z| \leq \|T\|\}$ and is in fact a bounded closed subset in the complex plane $\mathbb{C}$. We now claim that for a compact operator T the set $\sigma(T) \cap \{z : |z| \geq c\}$ is finite for $c > 0$. In other words, there are at most finitely many eigenvalues of T which are outside a disk around the origin. Yet another way to say this is to say that if $\sigma(T)$ has a limit point, then that point has to be 0.

To prove this, let us assume that $\{z : |z| \geq c\}$ is contains infinitely many eigenvalues. We will use this to show that T is not compact. Since this set is bounded, it contains a convergent sequence. Let (z_n) be a convergent sequence of eigenvalues of T which converges to w and all of these lie in $\{z : |z| \geq c\}$. Let v_n be the unit norm eigenvectors of T corresponding to z_n. Let V_n be the span of $v_1, \ldots, v_n$. This is a finite dimensional space, hence closed. As seen above, we can find x_n in $V_n \setminus V_{n-1}$ which are unit vectors so that $d(x_n, V_{n-1}) > 1/2$.

Exercise: Show that $(T - z_n\mathbf{1})x_n$ lies in V_{n-1}. (Hint: Use an expression of the form $x_n = \sum_{k=1}^{n} a_k v_k$ and the fact that v_k is an eigenvector with eigenvalue z_k.)

We will now show that $(1/z_n)Tv_n$ does not contain a convergent subsequence. We calculate, for $n > m$

$$\frac{1}{z_n}Tv_n - \frac{1}{z_m}Tv_m = v_n - v_m + \frac{1}{z_n}(T - z_n\mathbf{1})v_n - \frac{1}{z_m}(T - z_m\mathbf{1})v_m \in v_n + V_{n-1}$$

It follows that the left-hand side has norm at least $1/2$ by the choice of v_n.

Exercise: Show that $(1/z_n)v_n$ is a bounded sequence. (Hint: Show that the norm of all these vectors is bounded by $1/c$.)

Thus, we have bounded sequence whose image under T does not contain a convergent sequence. It follows that T is not compact.

We conclude that the spectrum of an infinite-dimensional compact operator is of the form $\{0\} \sqcup D$ where D is a collection of non-zero eigenvalues of the operator and it has no limit point except possibly 0. (D can be empty or finite and non-empty as well.) The element 0 *need not* be an eigenvalue. Finally, for each eigenvalue in D has a finite-dimensional space of eigenvalues.