Solutions to Quiz 6

1. Given an example of a linear operator $L : V \to W$ between normed linear spaces such that its graph is closed, but L is not continuous. (Hint: Consider the space V to be the space of polynomials with usual norm given as below. The operator $L : V \to V$ is differentiation.)

$$\|P\| = \sup_{t\in[0,1]} |P(t)|$$

Solution: We need to prove that the graph of L is closed. Consider a sequence (P_n, Q_n) in the graph which converges to (P, Q) in $V \times V$. This means:

1. Q_n is the derivative of P_n.
2. P_n converges to P in the norm given above.
3. Q_n converges to Q in the norm given above.

To prove that the graph is closed we need to show that Q is the derivative of P.

By the fundamental theorem of calculus, we have

$$P_n(t) = P_n(0) + \int_0^t Q_n(x)dx$$

We note that for t in $[0, 1]$, we have

$$\left|\int_0^t Q(x)dx - \int_0^t Q_n(x)dx\right| \le \int_0^t |Q(x) - Q_n(x)|dx \le t\|Q - Q_n\| \le \|Q - Q_n\|$$

It follows that $\int_0^t Q_n(x)dx$ converges to $\int_0^t Q(x)dx$. We also have

$$|P(t) - P_n(t)| \le \|P - P_n\|$$

It follows that $P_n(t)$ converges to $P(t)$ for all t in $[0, 1]$. Hence we obtain the limit

$$P_n(t) = P_n(0) + \int_0^t Q_n(x)dx \text{ converges to } P(0) + \int_0^t Q(x)dx$$

It follows that we have the equation for all t in $[0, 1]$.

$$P(t) = P(0) + \int_0^t Q(x)dx$$

Differentiating, we obtain that Q is the derivative of P at all t in $[0, 1]$. Hence, (P, Q) lies on the graph as required.

To see that L is *not* continuous, note that (by the Weierstrass approximation theorem) there is a sequence P_n of polynomials converging to $f(t) = |t - 1/2|$. Clearly, the derivatives of P_n cannot cannot converge in the given norm as the derivative of f does not exist at $t = 1/2$! By the Bernstein approach, one can even give P_n explicitly as follows

$$P_n(T) = \sum_{k=0}^{n} |(2k-n)/2n| \binom{n}{k} T^k (1-T)^{n-k}$$

It may be worthwhile to explicitly check that the derivatives have *unbounded* norm.