The Baire Category Theorem and Applications

There are a number of contexts where we would like a certain set-theoretic property to imply a more geometric, topological or analytic statement. For example, if $L : V \to W$ is a 1-1 onto map of Banach spaces and L is continuous, does this mean that it is an isomorphism? In other words, is it "automatic" that the inverse is continuous? What properties of a subset $Z \subset X \times Y$ will ensure that it is the graph is a continuous map from X to Y? In what cases can we assert that if a limit of functions exists point-wise then the limiting values give a nice function? To give an example of a different kind, if l_n is a sequence of lines in the plane, how can I show that there is a point in the plane that does not lie on any of these lines?

The Baire category theorem is a useful result in order to answer such questions. It states if U_n is a sequence of dense open sets in complete metric space (X, d), then the intersection of these sets is dense in X; in other words, the set $G = \cap_n U_n$ is dense in X.

To prove this, we will show that G meets *every* open set in X. In fact, it is enough to show that G meets the open ball $B_X(x, r) = \{y \mid d(x, y) < r\}$ for every x in X and $r > 0$. (In this section, given a normed linear space V we use $B_V(v, r)$ to denote the open ball consisting of all vectors w in W such that $\|w - v\| < r$.)

Exercise: For any s such that $0 < s < r$ show that the closure $\overline{B_X(x, s)}$ is contained in $B_X(x, r)$. (Hint: Note that if y_n satisfy $d(x, y_n) < s$ and y_n converge to z, then $d(x, z) \leq s$.)

Since U_1 is dense in X, the intersection $U_1 \cap B_X(x, r/2)$ is non-empty. Moreover, both these sets are open. Hence there is an x_1 and an $0 < r_1 \leq r/2$ so that $B_X(x_1, r_1)$ is contained in the intersection; in other words

$$B_X(x_1, r_1) \subset U_1 \cap B_X(x, r/2)$$

Exercise: Show that there is a sequence of elements x_n in X and r_n so that $0 < r_{n+1} \geq r_n/2$ so that

$$B_X(x_{n+1}, r_{n+1}) \subset U_{n+1} \cap B_X(x_n, r_n/2)$$

(Hint: Use induction on n and the fact that U_n is open and dense.)

Exercise: Show that x_{k+p} lies in $B_X(x_k, r_k/2)$ for all $p \geq 0$. (Hint: Use induction on p.)

Exercise: Show that (x_n) is a Cauchy sequence. (Hint: Use induction to prove that $r_n \leq r/2^n$.)

Since X is a complete metric space, it follows that (x_n) converges to a point y in X.

Exercise: Show that y lies in $\overline{B_X(x_k, r_k/2)}$ for any k. (Hint: Note that y is the limit of (x_{k+p}) for $p \geq 0$.)

As seen above, this means that y lies in $B_X(x_k, r_k)$ and thus in U_k for every k. In other words, y lies in G. We also note that y lies in $\overline{B_X(x, r/2)}$ since each x_k lies in this closed set. It follows that y lies in $B_X(x, r)$ as well. This proves what we want to prove.

In applications, this result is often used in the following form. Let X_n be a sequence of *closed* sets in a complete metric space (X, d) such that $X = \cup_n X_n$; in other words, X is the union of the sets X_n. We then claim that the interior of at least one X_n is non-empty. We can prove this by contradiction as follows.

Suppose that X_n has empty interior for all n. It follows that the complement $U_n = X \setminus X_n\$$ of X_n is open and dense in X for all n. It follows that $G = \cap_n U_n$ is dense in X. In particular, G is non-empty. However, $X \setminus G = \cup_n X_n = X$. So this is a contradiction.

Open Mapping Theorem

Given a bounded linear map $L : V \to W$ between Banach spaces, we show that, if L is onto then image of the unit ball under L contains an open ball in W. In other words, there is a $c > 0$ so that the open ball $B_W(0, c)$ of radius c in W is contained in the image $L(B_V(0, 1))$ of the unit ball in V.

First, we will show that $\overline{L\left(B_V(0, 1)\right)}$ contains $B_W(0, d)$ for a suitable $d > 0$. This will use the fact that W is complete and the Baire category theorem.

Secondly, we will use the completeness of V to show if $B_W(0, d)$ is contained in $\overline{L\left(B_V(0, 1)\right)}$ for a suitable d, then $L\left(B_V(0, 1)\right)$ contains $B_W(0, d/4)$.

To see the first part, we note that saying L is surjective is the same as saying that the union of the sets $L\left(B_V(0, n)\right)$, as n varies over positive integers, is all of W. Note that these are not closed sets, so the Baire category theorem does not apply directly. However, we note that W is also the union of the sets $\overline{L\left(B_V(0, n)\right)}$ as n varies.

Exercise: Show that there is a y in W and a $r > 0$ so that $\overline{L\left(B_V(0, 1)\right)}$ contains $B_W(y, r)$. (Hint: Use the Baire category theorem and note that $\overline{L\left(B_V(0, n)\right)}$ is the same as $n\overline{L\left(B_V(0, 1)\right)}$.)

Exercise: If y lies in $\overline{L\left(B_V(0, 1)\right)}$, then $-y$ also lies in $\overline{L\left(B_V(0, 1)\right)}$. (Hint: L is a linear operator.)

Exercise: Show that if $B_W(y, r)$ is contained in $\overline{L\left(B_V(0, 1)\right)}$, then $B_W(0, r)$ is contained in $\overline{2L\left(B_V(0, 1)\right)}$. Hint: Note that

$$\overline{L\left(B_V(0, 1)\right)} + \overline{L\left(B_V(0, 1)\right)} \subset \overline{L\left(B_V(0, 2)\right)}$$

It follows that $B_W(0, d)$ is contained in $\overline{L\left(B_V(0, 1)\right)}$ if $d = r/2$.

We now move to the second part where we assume that $B_W(0,d)$ is contained in $\overline{L\left(B_V(0,1)\right)}$.

Exercise: Given w in $B_W(0,d/2^n)$, show that there is a x in $B_V(0,1)$ such that $2^n w - L(x)$ lies in $B_W(0,d/2)$. (Hint: $2^n w$ lies in $B_W(0,d)$ which is in the closure of $L(B(0,1))$.)

Now start with any z in $B_W(0,d/4)$. Applying the above exercise once, we find $x_1 \in B_V(0,1)$ so that $4z - L(x_1)$ lies in $B_W(0,d/2)$. We now put $z_1 = z - L(x_1/4)$ and note that z_1 lies in $B_W(0,d/8)$. So we can apply the exercise once again to find x_2 in $B_V(0,1)$ so that $8z_1 - L(x_2)$ lies in $B_W(0,d/2$. We then put $z_2 = z_1 - L(x_2/8)$. We note that $z_2 = z - L(x_1/4 + x_2/8)$ and z_2 lies in $B_W(0,d/16$.

Repeating this, we find $z_n = z - L(x_1/4 + \cdots + x_n/2^{n+1})$ so that z_n lies in $B_W(0,d/2^{n+2})$; moreover, x_i lie in $B_V(0,1)$. Applying the exercise, we find x_{n+1} in $B_V(0,1)$ so that $2^{n+2}z_n - L(x_{n+1})$ lies in $B_W(0,d/2)$. We then put $z_{n+1} = z_n - L(x_{n+1}/2^{n+2})$. We again see that $z_{n+1} = z - L(x_1/4 + \cdots + x_n/2^{n+1} + x_{n+1}/2^{n+2})$, and z_{n+1} lies in $B_W(0,d/2^{n+3})$.

Continuing this way, we produce x_n in $B_V(0,1)$ for all n so that putting $z_n = z - L(x_1/4 + \cdots + x_n/2^{n+1})$ we have z_n lies in $B_W(0,d/2^{n+2})$.

Since V is a Banach space, the series $\sum_n x_n/2^{n+1}$ converges to an element x of $\overline{B_V(0,1/2)}$. Since L is continuous, we see that $L(x_1/4 + \cdots + x_n/2^{n+1})$ converges to $L(x)$ as n goes to infinity. On the other hand, we see that $L(x_1/4 + \cdots + x_n/2^{n+1})$ converges to z as n goes to infinity by construction. Hence, we see that $z = L(x)$. Now, $\overline{B_V(0,1/2)}$ is contained in $B_V(0,1)$. So, we see that z is in $L(B_V(0,1))$.

This completes the proof of the open mapping theorem.

Invertibility of continuous isomorphisms

Given a continuous linear operator $L : V \to W$ between Banach spaces which is one-to-one and onto. The usual theory about linear maps shows that there is a *linear* map $M : W \to V$ so that $M \circ L = \mathbf{1}_V$ and $L \circ M = \mathbf{1}_W$. Since L is continuous, the open mapping theorem shows that L is open. Hence, there is a $c > 0$ so that $B_W(0,c) \subset L(B_V(0,1))$; equivalently $M(B_W(0,c)) \subset B_V(0,1)$. This proves that M is bounded since $\|Mw\| \leq (1/c)\|w\|$ for all w in W.

In other words, a continuous linear operator between Banach spaces which is one-to-one and onto has a *continuous* inverse.

One can apply this as follows to a pair of norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V. Suppose that both norms make V into a Banach space. Moreover, let us assume that there is a positive constant C such that $\|v\|_1 \leq C\|v\|_2$ for all v in V. The identity map $(V, \|\cdot\|_2) \to (V, \|\cdot\|_1)$ is then a continuous one-to-one and onto linear operator. It follows that the identity map $(V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$ is *also* a continuous linear operator. In other words, we have a positive constant D such that $\|v\|_2 \leq D\|v\|_1$ for all v in V. In other words, the two norms are equivalent.

Closed Graph Theorem

Given normed linear spaces V and W, we can form the linear space $V \oplus W$ which consists of pairs (v, w) of vectors in V and W.

Exercise: Define $\|(v, w)\| = \|v\|_V + \|w\|_W$ and check that this gives a norm on $V \oplus W$.

Thus, $V \oplus W$ is a normed linear space in a natural way.

Exercise: The natural maps $i_V : V \to V \oplus W$ given by $i_V(v) = (v, 0)$ and $\pi_V : V \oplus W \to V$ given by $\pi_V((v, w)) = v$ are bounded linear maps.

The same applies to the natural maps i_W and π_W for W as well.

Exercise: Given a sequence (v_n, w_n) in $V \oplus W$, check that it is Cauchy if and only if v_n and w_n are separately Cauchy sequences.

It follows that $V \oplus W$ is a Banach space if V and W are Banach spaces. (The converse is also true. If $V \oplus W$ is a Banach space then each of V and W are Banach spaces. We will see one possible proof of this below!)

Exercise: A subspace W of a Banach space V acquires an induced normed linear space structure. Show that W is complete with respect to this norm if and only if W is closed in V.

One way to "construct" a linear subspace of $V \oplus W$ is as the graph Γ_L of a linear transformation $L : V \to W$; the space Γ_L consists of pairs of vectors of the form $(v, L(v))$. We note that there is a natural map $\tilde{L} : V \to \Gamma_L$ given by $v \mapsto (v, L(v))$. We note that $\pi_V \circ \tilde{L} = \mathbf{1}_V$, so that $\tilde{L}$ is one-to-one and onto. Further, we note that

$$\|v\|_V \leq \|v\|_V + \|L(v)\|_W = \|(v, L(v))\|$$

Equivalently, the map $(\pi_V)_{|\Gamma_L}$ is a continuous map $\Gamma_L \to V$. Now, *if* Γ_L is a closed subspace of $V \oplus W$, then, as seen above, it is *also* a Banach space. By the results of the previous section, we see that $\tilde{L} : V \to \Gamma_L$ is then continuous. We note that $L = \pi_W \circ \tilde{L}$ and π_W is continuous. Hence, L too is then continuous.

In other words, if the graph Γ_L of a linear transformation is a closed subspace of $V \oplus W$, then L is continuous.

Exercise: Show that if L is continuous, then the graph Γ_L is closed. (Hint: If $(v_n, L(v_n))$ is a Cauchy sequence in Γ_L, then v_n converges to a v in V.)

This equivalence between the continuity of the linear transformation L and the closed-ness of its graph Γ_L is called the closed graph theorem.

Uniform Boundedness and applications

Suppose that we are given a collection $L_i : V \to W$ for $i \in I$ of continuous linear transformations. Further, assume that for every v in V, the sequence $\{L_i(v) : i \in I\}$ in W is bounded. We claim that the collection of norms $\{\|L_i\| : i \in I\}$ is bounded.

To prove this consider the subset X_k of V which consists of all those v in V such that $\|L_i(v)\|_W \leq k$ for all $i \in I$. By assumption, we see that V is the union of X_k.

Exercise: Show that X_k is a closed subset of V. (Hint: It is the intersection of the closed sets $L_i^{-1}(\overline{B_W(0,k)})$ as i varies over I.)

By the Baire category theorem, if V is a Banach space, then at least one X_k has a non-empty interior. In other words, there is a v in V and an $r > 0$ so that $B_V(v,r)$ is contained in X_k. This means that

$$\|L_i(x)\| \leq \|L_i(v+x)\| + \|L_i(v)\| \leq 2k \text{ if } \|x\| < r$$

It follows that $\|L_i\| < 2k/r$ for all n and for all $i \in I$..

One case when we obtain a bounded collection is when we have $L_n(v)$ is a Cauchy sequence or a convergent sequence. So suppose that $L_n(v)$ converges some $L(v)$ for each v in V.

Exercise: Show that L is a linear operator from V to W. (Hint: Each L_n is linear.)

Exercise: Show that L is a bounded linear operator from V to W. (Hint: Check that the uniform bound for $\|L_n\|$ is also a bound for the norm of L.)

In other words, if a sequence of bounded linear operators converges *pointwise* to an operator L, then it is also a bounded linear operator.

Exercise: Let B be a subset of V so that for every linear functional $f : V \to \mathbb{C}$, the image of B is bounded. Show that B is a bounded subset of V. (Hint: Apply uniform boundedness to the collection B of linear operators on V^*.)