## Solutions to Quiz 5

1. Let $\ell_2$ denote the usual space of square summable series of complex numbers.

   Consider the linear operator $T : \ell_2 \to \ell_2$ defined by

$$T((a_1, a_2, \ldots, a_n, \ldots)) = (a_1, a_2/2, \ldots, a_n/n, \ldots)$$

   Let $\mathbf{1}$ denote the identity operator.

(1 mark) (a) What is $\|T\|$?

(1 mark) (b) Is the map $T$ invertible?

(1 mark) (c) Is the map $(1/2)T - \mathbf{1}$ invertible?

(1 mark) (d) If $z$ is a complex number which is *not* real, then is $T - z\mathbf{1}$ invertible?

(1 mark) (e) Is there a real number $a$ in $[0, 1]$ such that $T - a\mathbf{1}$ is invertible?

**Solution:** Much of the following is in the last section of the notes on "Invertible Operators and Spectrum".

Given a sequence $(b_n)$, the operator $P : \ell_2 \to \ell_2$ defined by

$$P((a_1, a_2, \ldots, a_n, \ldots)) = (b_1 a_1, b_2 a_2, \ldots, b_n a_n, \ldots)$$

is a bounded linear operator if and only if $\sup_{n=1}^{\infty} |b_n|$ is bounded. Now the *linear* inverse is given by

$$Q((a_1, a_2, \ldots, a_n, \ldots)) = (a_1/b_1, a_2/b_2, \ldots, a_n/b_n, \ldots)$$

For $P$ to be invertible, $Q$ has to be bounded. Hence, it is necessary and sufficient that $\sup_{n=1}^{\infty} |1/b_n|$ is bounded. This is the same as the condition $\inf_{n=1}^{\infty} |b_n| > 0$. Now we can easily answer the last four parts.

We have $\inf_{n=1}^{\infty} 1/n = 0$. So $T$ is *not* invertible.

We have $\inf_{n=1}^{\infty} |\frac{1}{2n} - 1| \neq 0$. So $(1/2)T - \mathbf{1}$ is invertible.

We have $\inf_{n=1}^{\infty} |1/n - z| \geq |y|$ where $y$ is the imaginary part of $z$. So if $y \neq 0$, then this is invertible. In other words, if $z$ is *not* real, the $T - z\mathbf{1}$ is invertible.

Take a number like $a = 2/3$ which is different from $1/n$ for all $n$ and not 0. Then $\inf_{n=1}^{\infty} |1/n - 2/3| = 1/6$, so $T - (2/3)\mathbf{1}$ is invertible.

Finally, to calculate the norm, we note that

$$\|A((a_1, \ldots, a_n, \ldots))\|_2^2 = \sum_{n=1}^{\infty} |a_n b_n|^2 \leq \sup_{n=1}^{\infty} |b_n|^2 \sum_{n=1}^{\infty} |a_n|^2 \leq \sup_{n=1}^{\infty} |b_n|^2 \|(a_n)\|_2^2$$

So $\|A\| \leq \sup_{n=1}^{\infty} |b^n|$. On the other hand we have

$$P((0, \ldots, 0, \overset{n}{\check{1}}, 0, \ldots)) = (0, \ldots, 0, \overset{n}{\check{b}_n}, 0, \ldots)$$

and

$$\|(0, \ldots, 0, \overset{n}{\check{c}}, 0, \ldots)\|_2 = |c|$$

Hence $\|P\| \geq |b_n|$. It follows that $\|P\| = \sup_{n=1}^{\infty} |b_n|$. Thus, $\|T\| = 1$.