Solutions to Quiz 4

1. Let ℓ_2 denote the usual space taken as an inner product space with the $\|\cdot\|_2$ norm and the associated inner product. Let $\mathbb{R}^N$ be the subspace consisting of vectors of the form $(a_1, \ldots, a_N, 0, \ldots)$; i. e. the vectors which have component 0 beyond the N-th component. Let $\mathbb{R}^\infty$ be the union of $\mathbb{R}^N$ over all N.

   Consider the linear functional $f : \mathbb{R}^\infty \to \mathbb{R}$ defined on $\mathbb{R}^N$ by

$$f((a_1, \ldots, a_N, 0, \ldots)) = \sum_{n=1}^{N} a_n/n$$

(1 mark) (a) Find the vector v_N in $\mathbb{R}^N$ such that $f(a) = \langle a, v_N \rangle$ for every a in $\mathbb{R}^N$.

> **Solution:** The vector $v_N = (1, 1/2, \ldots, \overset{N}{\check{1/N}}, 0, \ldots)$ has this property. It is the unique such vector.

(1 mark) (b) What is $\|v_N\|_2$?

> **Solution:** The norm $\|v_N\| = \sqrt{\sum_{n=1}^{N} (1/n)^2}$.

(1 mark) (c) Does the sequence v_N converge in $\mathbb{R}^\infty$ with respect to $\|\cdot\|_2$?

> **Solution:** No. The limit does not lie in $\mathbb{R}^\infty$ since all components of the limit must be non-zero.

(1 mark) (d) Does the sequence v_N converge in ℓ_2?

> **Solution:** Yes. The difference $v_M - v_N$ has norm $\sqrt{\sum_{n=N+1}^{M} 1/n^2}$ which is small for N, M sufficiently large. In fact the limit is the vector
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> $$v = (1, 1/2, 1/3, \ldots)$$
>
> For the next part note that, $\|v_n\|$ converges to $\|v\| = \|f\|$ as n goes to infinity by the Riesz representation theorem.

(1 mark) (e) Find a sequence w_n of *unit* vectors in ℓ_2 such that $f(w_n)$ goes to $\|f\|$ as n goes to infinity.

> **Solution:** If we take the vector $w_n = v_n/\|v_n\|_2$, then w_n is a unit vector. Moreover,
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> $$f(w_n) = \langle w_n, v_n \rangle = \frac{\langle v_n, v_n \rangle}{\|v_n\|_2} = \|v_n\|_2$$

This converges to $\|f\|$.