## Riesz' Representation Theorem

Given a vector $b$ in $H$, we can define a linear functional $h_b : H \to \mathbb{R}$ by defining $h_b(a) = \langle a, b \rangle$. We have seen that this gives a norm-preserving map $\lambda : H \to B(H, \mathbb{R})$. In this section we will show that this map is onto.

**Exercise**: Given a linear functional $h : \mathbb{R}^n \to \mathbb{R}$ show that there is a vector $b$ in $\mathbb{R}^n$ such that $h(a) = \langle a, b \rangle$, where this is the usual inner product on $\mathbb{R}^n$. (Hint: Take the $n$-coordinate of $b$ to be $h(e^{(k)})$ where $e^{(1)}, \ldots, e^{(n)}$ is the standard basis of $\mathbb{R}^n$.

### Separable Hilbert Spaces

We could try to generalise this approach to infinite dimensions, if we had something like the standard basis. For example, suppose that $H$ contains a countable dense set $D$. Let $B$ be a maximal linearly independent subset of $D$. Apply the Gram-Schmidt process to $B$ to obtain a sequence of vectors $e^{(n)}$ with the property that $\langle e^{(n)}, e^{(m)} \rangle = \delta_{m,n}$ is 1 if $m = n$ and 0 otherwise (an orthonormal sequence). In what sense can we say that this is a "basis" of $H$?

The orthonormal basis gives a linear map $e : \mathbb{R}^\infty \to H$ in the obvious way: a vector $a = (a_n)$ in $\mathbb{R}^N$ is mapped to $e(a) = \sum_{n=1}^N a_n e^{(n)}$.

**Exercise**: Check that

$$\|e(a)\| = \sum_{n=1}^{N} |a_n|^2 = \|a\|_2$$

In other words, the linear map $e$ preserves norms if we use the norm $\|\cdot\|_2$ on $\mathbb{R}^\infty$. Thus, the map $e$ extends to a norm-preserving map on $\ell_2$ which is the completion of $\mathbb{R}^\infty$.

On the other hand, $D$ is dense in $H$, and $D$ is contained in the linear span of $B$ which is also the linear span of the sequence $e^{(n)}$ which, in turn is $\mathbb{R}^\infty$. In other words, $e : \ell_2 \to H$ is norm-preserving and onto; hence, it is an isomorphism of normed linear spaces.

**Exercise**: If $f : H_1 \to H_2$ is a norm-preserving isomorphism of Hilbert spaces, then show that the inner-product of $a$ and $b$ in $H_1$ is the same as the inner-product of $f(a)$ and $f(b)$ in $H_2$. (Hint: Use the formula for the inner product in terms of the norm!)

Continuous linear functionals on $\ell_2$ can be identified by their restriction to $\mathbb{R}^\infty$. In turn, linear functionals of $\mathbb{R}^\infty$ can be identified with sequences. In an earlier section, we used continuity with respect to $\|\cdot\|$ to identify the sequences arising from continuous linear functionals with $\ell_2$. Thus, this proves Riesz representation theorem in this case.

Since *most* Hilbert spaces of interest turn out to be separable Hilbert spaces, this is an adequate argument. However, there are some nice ideas in the next proof of Riesz Representation theorem which are of independent interest.

**Orthogonal projection**

For $n > m$, consider the subspace $\mathbb{R}^m$ of $\mathbb{R}^n$ which consists of vectors where the last $n - m$ co-ordinates are 0.

**Exercise**: Given a vector $v = (x_1, \ldots, x_n)$, put $\pi(v) = (x_1, \ldots, x_m, 0, \ldots, 0)$ and show that

$$\|v - \pi(v)\| = \text{ distance from } v \text{ to } \mathbb{R}^m \ = \inf_{w \in \mathbb{R}^m} \|v - w\|$$

More generally, if $W$ is a *finite* dimensional subspace of an inner-product space $V$, then:

**Exercise**: Given a vector $v$ in an inner product space $V$ and a finite dimensional subspace $W$ of $V$, there is a vector $\pi(v)$ in $W$ such that

$$\|v - \pi(v)\| = \text{ distance from } v \text{ to } W \ = \inf_{w \in W} \|v - w\|$$

(Hint: Extend an orthonormal basis of $W$ to an orthonormal basis of $W + \mathbb{R}v$ and apply the previous exercise to the corresponding copy of $\mathbb{R}^n$ inside $V$.)

**Exercise**: Given a subspace $W$ of an inner product space $V$ and vectors $v \in V$ and $w_0 \in W$ such that

$$\|v - w_0\| = \text{ distance from } v \text{ to } W \ = \inf_{w \in W} \|v - w\|$$

Show that $\langle v - w_0, w \rangle = 0$ for all $w$ in $W$.

**Exercise**: Given a subspace $W$ of an inner product space $V$ and vectors $v \in V$ and $w_0 \in W$ such that $\langle v - w_0, w \rangle = 0$ for all $w$ in $W$, show that

$$\|v - w_0\| = \text{ distance from } v \text{ to } W \ = \inf_{w \in W} \|v - w\|$$

So, for a finite dimensional subspace $W$, the vector $\pi(v)$ is the base of a "perpendicular drop" from $v$ to $W$. In order to extend this to infinite dimensional $W$'s, we need to understand the notion of an "approximate perpendicular". First we need a simple application of the parallelogram identity:

**Exercise**: Given vectors $v$ and $w$ such that $\|v\|^2$, $\|w\|^2$ and $\|\frac{v+w}{2}\|^2$ are within $\epsilon$ of each other (i.e. all lie in an interval of length $\epsilon$), we have $\|v - w\|^2 < 4\epsilon$. (Hint: The parallelogram identity can be stated as

$$\|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2 - 4\|\frac{v + w}{2}\|^2$$

Now we can estimate the right-hand side.)

A similar idea can be used for the following:

**Exercise**: Given vectors $v$ and $w$ such that $\|v\|^2$, $\|w\|^2$ and $\langle v, w\rangle$ are within $\epsilon$ of each other (i.e. all lie in an interval of length $\epsilon$), we have $\|v-w\|^2 < 2\epsilon$. (Hint: We have the identity

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\langle v, w\rangle$$

Now we can estimate the right-hand side.)

For $V$ an inner product space, $W$ a subspace and $v$ a vector in $V$, let $d = d(v, W) = \inf_{w\in W}\|v-w\|$.

**Exercise**: For every $n \geq 1$ show that there is a vector $w_n$ in $W$ such that $\|v-w_n\|^2 < d^2 + 1/n$.

Given $\epsilon > 0$ we take $N$ so that $\epsilon^2 > 4/N$ and claim that for all $n, m >\geq N$ we have $\|w_n - w_m\| < \epsilon$. We first note that $(w_n + w_m)/2$ lies in $W$ so that

$$\|(v-w_n) + (v-w_m)\|^2 = 4\|v - \frac{w_n + w_m}{2}\| > 4d^2$$

Using the parallelogram identity

$$\begin{aligned}
&\|w_n - w_m\|^2 = \\
&\qquad 2\|v-w_n\|^2 + 2\|v-w_m\|^2 - 4\|v - \frac{w_n+w_m}{2}\|^2 \\
&= 2\left(\|v-w_n\|^2 - \|v - \frac{w_n+w_m}{2}\|^2\right) + 2\left(\|v-w_m\|^2 - \|v - \frac{w_n+w_m}{2}\|^2\right) \\
&\qquad \leq \frac{2}{n} + \frac{2}{m} \leq \frac{4/N}{<}\epsilon^2
\end{aligned}$$

This proves that the sequence $w_n$ is a Cauchy sequence in $W$. In summary, given a vector $v$ in an inner product space $V$ and a subspace $W$, we have produced a Cauchy sequence $w_n$ of vectors in $W$ such that $\|v - w_n\|$ decreases to $d(v, W) = \inf_{w\in W}\|v-w\|$.

Now, assume that $V$ is a Hilbert space and that $W$ is a *closed* subspace. In that case, the Cauchy sequence $w_n$ converges to a vector $\tilde{v}$ and this vector lies in $W$ (since $W$ is closed). This vector has the property that $\|v-\tilde{v}\| = d(v, W)$. As seen above, this means that for $w$ another vector in $W$ we have $\langle v - \tilde{v}, w\rangle = 0$. In other words, $\tilde{v}$ is the "orthogonal projection" of $v$ to $W$.

Conversely, we have seen that if $\tilde{v} \in W$ is such that $\langle v - \tilde{v}, w\rangle = 0$ for all $w$ in $W$, then $\|v - \tilde{v}\| = d(v, W)$. In fact, if $w \in W$ is any *non-zero* vector then

$$\|v - (\tilde{v} + w)\|^2 = \|v - \tilde{v}\|^2 + \|w\|^2 > \|v - \tilde{v}\|^2$$

So this vector is *uniquely* determined. This allows us to define a map $\pi : V \to W$ by sending each vector $v$ to the vector $\tilde{v}$ as constructed above (when $V$ is a Hilbert space). We note that if $v$ and $v'$ are vectors in $V$, then $v - \pi(v)$ and $v' - \pi(v')$ are orthogonal to $W$. Hence, so is $(v + v') - (\pi(v) + \pi(v'))$. By the uniqueness of the orthogonal projection, we see that $\pi(v + v') = \pi(v) + \pi(v')$. Similarly,

**Exercise**: Show that $\pi(a \cdot v) = a \cdot \pi(v)$.

In other words, $\pi$ is a linear map. Moreover, by construction, $\|v\|^2 = \|\pi(v)\|^2 + \|v - \pi(v)\|^2$ so that $\|\pi\| \leq 1$. Thus, $\pi$ is a continuous linear functional. We have thus proved:

*Given a closed subspace $W$ of a Hilbert space $H$, there is a bounded linear functional $\pi_W : H \to H$ with image $W$ such that for each vector $v$, the vector $v - \pi_W(v)$ is orthogonal to $W$.*

By the uniqueness of $\pi(v)$, we can easily see that $\pi(w) = w$ for all $w \in W$. It follows that (if $W \neq 0$!) then $\pi_W$ has norm 1.

### Linear functionals

Given a continuous linear functional $f : V \to \mathbb{R}$ on an inner product space $V$. We first want to "approximate" it by inner product with a vector. How does one find a $w$ so that the linear functional associated with $w$ is "close" to $f$?

**Exercise**: Given any $\epsilon > 0$, there is a unit vector $u_\epsilon$ so that $f(u_\epsilon) > \|f\| - \epsilon$.

Putting $w_\epsilon = f(u_\mu)u_\mu$ where $\mu$ is chosen suitably, we can prove:

**Exercise**: Given any $\epsilon > 0$, there is a vector $w_\epsilon$ so that $f(w_\epsilon)$ and $\|w_\epsilon\|^2$ lie in the interval $\left[\|f\|^2 - \epsilon, \|f\|^2\right]$.

Let $W$ be a finite dimensional subspace of $V$ which contains $w_\epsilon$. By an earlier exercise we know that there is a vector $w_1$ in $W$ such that $f(w) = \langle w, w_1 \rangle$ for all $w$ in $W$. We then have $\|w_1\| \leq \|f\|$. Thus, $\|w_1\|^2$, $\|w_\epsilon\|^2$ and $\langle w_\epsilon, w_1 \rangle$ *all* lie in the interval $\left[\|f\|^2 - \epsilon, \|f\|^2\right]$. We can thus apply the exercise of the previous subsection to show that $\|w_1 - w_\epsilon\|^2 < 4\epsilon$. Since $f(w) = \langle w, w_1 \rangle$ for $w$ in $W$, this also shows that

$$|f(w) - \langle w, w_\epsilon \rangle| < 4\epsilon\|w\| \text{ for } w \in W$$

Since any vector $v$ in $V$ lies in some such finite dimensional $W$, it follows that $f$ is approximated to an error of at most $4\epsilon$ by $w_\epsilon$.

**Exercise**: Complete the argument above to show that for any $v$ in $W$ we have

$$|f(v) - \langle v, w_\epsilon \rangle| < 4\epsilon\|v\|$$

We can now create a sequence $v_n = w_{1/n}$ of vectors in $V$ and show that it is a Cauchy sequence as before. If $V$ is a Hilbert space, then this sequence converges to a vector $v_f$ for which $f(v) = \langle v, v_f \rangle$ for every $v$ in $V$.

A less "constructive" and more conceptual argument is as follows. Given a continuous linear functional $f : H \to \mathbb{R}$, we want a vector $v_f \in H$ such that $f(v) = \langle v, v_f \rangle$ for all $v$ in $H$. If $f = 0$ there is nothing to prove. Otherwise, let $v_0$ be such that $f(v_0) \neq 0$. Secondly, let $W$ be the collection of vectors $w \in H$ such that $f(w) = 0$. Since $W$ is closed ($f$ is continuous!), there is a (unique) vector $v_1 \in W$ such that $v_0 - v_1$ is orthogonal to $W$. Note that $f(v_0 - v_1) = f(v_0)$ since $f(v_1) = 0$. Replacing $v_0$ by $v_0 - v_1$, we may assume that $v_0$ is orthogonal to $W$. Secondly, we note that $v_0 \neq 0$ since $f(v_0) \neq 0$. So we note that

$$f(v_0) = \langle v_0, \frac{f(v_0)}{\|v_0\|^2} \cdot v_0 \rangle$$

So if $v_f = \frac{f(v_0)}{\|v_0\|^2} \cdot v_0$, then we can prove:

**Exercise**: Check that $f(v) = \langle v, v_f \rangle$ for every $v$ in $V$. (Hint: Write $v$ as $w + b \cdot v_0$ for $w$ in $W$ using the orthogonality of $W$ and $v_0$.)