Solutions to Quiz 2

1. Let $\mathbb{C}^\infty$, $\mathcal{C}_0$ and $\mathcal{C}$ denote the usual spaces of sequences with the norm $\|\cdot\|_\infty$. Define a linear functional:

$$f : \mathbb{C}^\infty \to \mathbb{C} \text{ by } f((a_n)) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

(1 mark) (a) What is the norm of f?

> **Solution:** If $\sup_n |a_n| = \alpha$, then
>
> $$|f((a_n))| \leq \sum_{n=1}^{\infty} \frac{\alpha}{2^n} = \alpha$$
>
> So $\|f\| \leq 1$. On the other hand, if we take $a^{(n)} = (1, \ldots, \overset{n}{\check{1}}, 0, \ldots)$, then $\|a^{(n)}\| = 1$ and $f(a^{(n)}) = \sum_{k=1}^{n} 1/2^k$. So
>
> $$\|f\| \geq \sum_{k=1}^{n} \frac{1}{2^k}$$
>
> for all n. It follows that $\|f\| = 1$.

(2 marks) (b) Let g be a continuous extension of f to $\mathcal{C}_0$. What is $g(b)$ where $b = (1, 1/2, 1/3, \ldots)$?

> **Solution:** The sequence $b^{(n)} = (1, 1/2, \ldots, 1/n, 0, \ldots)$ consists of elements of $\mathbb{C}^\infty$ which converge to b in $\|\cdot\|_\infty$.
> Since g is continuous $g(b)$ is the limit of $g(b^{(n)})$. This shows that
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> $$g(b) = \sum_{n=1}^{\infty} \frac{1}{n2^n} = -\log(1 - 1/2) = \log(2)$$

(2 marks) (c) Let h be a continuous extension of f to $\mathcal{C}$. What is $h(c)$ where $c = (1, 1, 1, \ldots)$?

> **Solution:** Since $\mathcal{C}_0$ is closed in $\mathcal{C}$ and $\mathcal{C} = \mathcal{C}_0 + \mathbb{C} \cdot c$, we have:
>
> 1. A continuous map $\pi : \mathcal{C} \to \mathcal{C}_0$ such that $\pi(c) = 0$ and $\pi(a) = a$ for $a \in \mathcal{C}_0$.
> 2. A continuous map $t : \mathcal{C} \to \mathbb{C}$ so that $t(c) = 1$ and $t(\mathcal{C}_0) = \{0\}$.
>
> More explicitly, we can take $t(a) = \lim_{n\to\infty} a_n$ (Check that this is continuous!) and $\pi(a) = b$ where b is the sequence $(a_n - t(a))$ (Check that this is continuous!).

Thus, for *any* complex number z we can take $h = g \circ \pi + z \cdot t$. Then $h(a) = g(a)$ for $a \in \mathcal{C}_0$ and $h(c) = z$. So there is no well-defined *unique* extension of g.